蛙鸣

第65期

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1981.6-2022.6

中国科大学生数学杂志

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马志明

2021年6月4日

卷首语

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在我们每一个人的心中蛰伏

时光荏苒，当年的小作者们已然成为我们身边的师长，成为了杰出的数学工作者，《蛙鸣》见证了他们的成长。2021 年 6 月 20 日，时隔十年我们复刊发布了第 64 期《蛙鸣》。它是属于我们每个人的，我们一起“咕呱咕呱”，一起“哇啦哇啦”，汇成盛夏。

2022 年仲夏，从同学们的一场冒险、一次探索、一篇习作或阅读笔记，到名家名作的译稿，前沿研究的介绍，吟一篇诗词歌赋，问题提出又解答，这所有的一切都将见证我们的成长。

第 65 期《蛙鸣》编辑部
2022 年 6 月 6 日

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稻花香里说丰年，听取蛙声一片。

——《西江月 · 夜行黄沙道中》

Isoperimetric Inequality: Its Origin, Proof and Development

Li Xuanyu

Abstract

In this paper, we discuss the isoperimetric inequality. We will first give the proof of isoperimetric inequality on $\mathbb{R}^2$, then list the results of general cases, including the isoperimetric inequality on higher dimensional Euclidean spaces, manifolds with positive curvature and minimal submanifolds.

1 The Isoperimetric Inequality on Euclidean Spaces

1.1 Plane Case

From ancient Greece on, people had been long considering the following problem:

Of all plane closed curves with equal perimeters, which one bounds the largest area?

For example, Zenodorus[1] knew a circle has greater area than any polygon with the same perimeter. His work was summarized in Pappu's[2] *Collection*. Although the circle is the obvious answer to the problem, no one knew how to prove it for a long period. One reason was that everybody assumed the answer was correct since it was too obvious, just as the fundamental theorem of algebra. Another reason was that without calculus, it would be hard to talk about the area of arbitrary domain.

The first progress was made by Steiner[3] in 1838. By following argument, he showed that the only possible solution was circle[4]:

(1) If a domain is not convex, then there is a convex domain with same perimeter but greater area;

(2) If a domain is not symmetry, then we can find a symmetry domain which has same perimeter but greater area.

However, Steiner's proof was flawed. Weierstrass[4] pointed out that Steiner only showed the UNIQUENESS of the solution, but he did not prove the EXISTENCE. Weierstrass further constructed a counterexample. Later in 1879, by his original variational method, Weierstrass gave out the first rigorous proof of the isoperimetric inequality[5].

After Weierstrass, numerous proofs of the isoperimetric inequality was found. For example,in 1901 Hurwitz[5] proved it by Wirtinger's inequality[6]. It is said that there are

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稿件信息: 3 月 17 日来稿, 4 月 26 日第一次修改, 5 月 18 日接受. 稿件编辑: 宋晨楷.

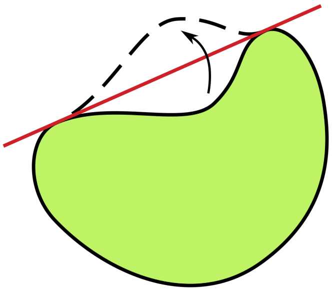
[1] About 200-140 BC.

[2] About 290-350 AD, Greek mathematician.

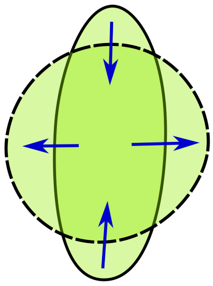
[3] Jacob Steiner, 1796.3-1863.4, Swiss geometer.

[4] Karl Weierstrass, 1815.10-1897.1, German mathematician, known for establishing formal definitions of calculus.

[5] Adolf Hurwitz, 1859.3-1919.11, German mathematician, known for studies of Riemann surfaces.



(a) Flip to get a convex domain which has greater area.



(b) Tilt to get a symmetry domain with greater area.

图 1.1: Steiner's argument. Source: Wikipedia

more than 100 different approaches to the isoperimetric inequality[6]. Here we give another proof which was found by Schmidt[7]in 1943[7].

Theorem 1.1. *Suppose C is a simple closed curve with length L, and A is the area it bounds, then*

$$L^2 \geqslant 4\pi A.$$

Moreover, the equality holds if and only if C is a circle.

Proof. We enclose C between two parallel lines l and l', such that C lies between l and l' and is tangent to them at the point P and P' respectively. Then construct a circle $\bar{C}$ tangent to l, l' at Q, Q' respectively. Denotes its radius by r and take its center to be the origin of a coordinate system. See figure 2.

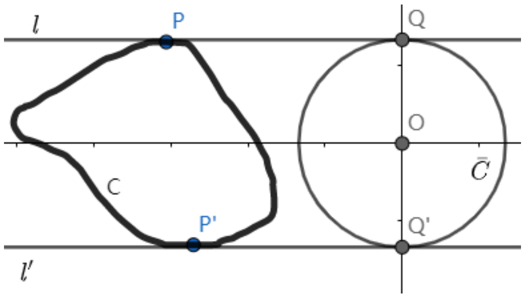


图 1.2: Curve C and $\bar{C}$.

Let $\boldsymbol{r}(s) = (x(s), y(s)), 0 \leqslant s \leqslant L$ be a parameterization of C, parameterized by arc length. Take s such that P, P' have parameter $s = 0, s_0$ respectively. Suppose $\bar{\boldsymbol{r}} = (x(s), \bar{y}(s))$ is a parameterization of $\bar{C}$, such that

$$\bar{y}(s) = \begin{cases} \sqrt{r^2 - x^2(s)}, & 0 \leqslant s \leqslant s_0; \\ -\sqrt{r^2 - x^2(s)}, & s_0 < s \leqslant L. \end{cases}$$

The area bounded by C is $A = \int_C x dy = \int_0^L xy' ds$. The area bounded by $\bar{C}$ is $\bar{A} = \pi r^2 = -\int_{\bar{C}} y dx = -\int_0^L \bar{y} x' ds$. Adding these two equations, we have

$$2\sqrt{\pi A} r \leqslant A + \pi r^2 = \int_0^L (xy' - \bar{y}x') ds \leqslant \int_0^L \sqrt{(x^2 + \bar{y}^2)(y'^2 + x'^2)} ds = Lr, \tag{1.1}$$

[6]Said prof. Zhang Xi, in his differential geometry class.

[7]Erhard Schmidt, 1876.1-1959.12, German mathematician, the Gram-Schmidt process in linear algebra is named after him.

since $x^2 + \bar{y}^2 = r^2$ and $x'^2 + y'^2 = 1$. We have proved the isoperimetric inequality.

Now, suppose the equality holds. By the condition of first inequality in (1.1), we have $A = \pi r^2$ and $L = 2\pi r$. Moreover, by the second inequality in (1.1), we have $x/y' = -\bar{y}/x' = \pm\sqrt{x^2+\bar{y}^2}/\sqrt{x'2+y'^2} = \pm r$. In particular, $x = \pm r y'$. At last, we exchange the x-axis and y-axis to get $y = \pm r x'$, hence

$$x^2 + y^2 = r^2.$$

□

1.2 Higher Dimensional Cases

The isoperimetric inequality can be also generalized into higher dimensional Euclidean spaces. In this case, the isoperimetric inequality is written as

$$|\partial D| \geqslant n|B^n|^{\frac{1}{n}}|D|^{\frac{n-1}{n}}, \tag{1.2}$$

where $|\cdot|$ denotes an appropriate dimensional measure of a set. For example, $|\partial D|$ denotes the $(n-1)$-dimensional measure of ∂D. And B^n is the unit ball in $\mathbb{R}^n$ centered at origin, D is a compact domain in $\mathbb{R}^n$.

At first, mathematicians proved (1.2) under some assumption on the ∂D. For instance, Schmidt proved (1.2) when D was rotationally symmetric in 1949[8] and Hadwiger[8] when ∂D was smooth in 1957[9]. The most general case, say, D and ∂D are only measurable, is a corollary of Brunn-Minkowski inequality. This work was a milestone in geometric measure theory and was done by Federer in 1969[10]. USTC graduate Cao Hongyi gave a detailed proof of the Brunn-Minkowski inequality in the Warming, vol. 64. We give another approach to (1.2) below, which depends on the solvability of Neumann problem and hence requires ∂D to be smooth. It was given by Cabré in 2008[11].

Theorem 1.2. *Suppose D is a bounded domain in $\mathbb{R}^n$ with ∂D smooth. Then*

$$|\partial D| \geqslant n|B^n|^{\frac{1}{n}}|D|^{\frac{n-1}{n}}.$$

Moreover, the equality holds if and only if D is a ball.

Proof. Consider the Neumann problem on D:

$$\begin{cases} \Delta u = \frac{|\partial D|}{|D|} & in \quad D, \\ \frac{\partial u}{\partial \eta} = 1 & on \quad \partial D. \end{cases}$$

where η denotes the outer normal vector of ∂D. $\int_D \Delta u = \int_{\partial D} \frac{\partial u}{\partial \eta}$ ensures the existence of u. Consider the lower contact set of u, defined by

$$\Gamma_u := \{x \in D | u(y) \geqslant u(x) + \nabla u(x)\cdot(y-x), \forall y \in \bar{D}\}.$$

Geometrically speaking, Γ_u is the set of all points x such that the tangent space of u in x lies under the graph of u. As a result, u is convex on Γ_u.

Next we prove $B^n \subset \nabla u(\Gamma_u)$. In fact, take arbitrary ξ such that $|\xi| < 1$. Consider $w(x) = u(x) - \langle \xi, x\rangle$. $\frac{\partial w}{\partial \eta} = 1 - \langle \xi, \eta\rangle > 0$ on ∂D, hence w cannot attain its minimum on ∂D.

[8] Hugo Hadwiger, 1908.12-1981.10, Swiss mathematician.

Suppose $\bar{x} \in D$ satisfies $w(\bar{x}) = \inf_{\bar{D}} w$. Then from $\nabla w(\bar{x}) = 0$ we get $\xi = \nabla u(\bar{x})$ and from $w(y) \geqslant w(\bar{x}), \forall y \in \bar{D}$ we get $\bar{x} \in \Gamma_u$.

As a result,

$$|B^n| \leqslant |\nabla u(\Gamma_u)| \leqslant \int_{\Gamma_u} \det D^2 u.$$

Since u is convex on Γ_u, the eigenvalues of $D^2 u$ are positive on Γ_u. Apply geometric-arithmetic mean inequality to these eigenvalues, we get

$$\det D^2 u \leqslant \left(\frac{\mathrm{tr} D^2 u}{n}\right)^n = \left(\frac{\Delta u}{n}\right)^n = \left(\frac{|\partial D|}{n|D|}\right)^n.$$

Hence

$$|B^n| \leqslant \int_{\Gamma_u} \left(\frac{|\partial D|}{n|D|}\right)^n = \left(\frac{|\partial D|}{n|D|}\right)^n |\Gamma_u| \leqslant \left(\frac{|\partial D|}{n|D|}\right)^n |D|$$

as we want. If the equality holds, then Γ_u is dense in D by $|\Gamma_u| = |D|$. Similarly B^n is dense in $\nabla u(\Gamma_u)$. However, Γ_u is closed, so $\Gamma_u = D$, then $\nabla u(D) = B^n$ which means $|\nabla u| < 1$ on D. Since the mean inequality is a equality on Γ_u, we have $D^2 u = \lambda I$ on $\Gamma_u = D$, where $\lambda = \frac{|\partial D|}{n|D|}$. As a consequence, $u = \lambda |x - x_0|^2/2 + c$. Putting these facts together, we get

$$D \subset \{x : |\partial D||x - x_0| < n|D|\}.$$

Compare the volume of the both side, we know they are actually the same set, which means D is a ball. □

It is worth mentioning that this method, say, proving isoperimetric inequality via elliptic equations and lower contact set, is so-called ABP method. Trudinger first used this idea to prove the isoperimetric inequality by applying ABP estimate to the following Monge-Ampère equation[12]:

$$\begin{cases} \det D^2 u = \chi_D & in \quad B_R, \\ u = 0 & on \quad \partial B_R. \end{cases}$$

ABP method is powerful in proving some geometric inequalities. Later in 2016, together with Ros-Oton and Serra, Cabré proved a weighted isoperimetric inequality in Euclidean space[13]. Besides these results, we will mention ABP method again in the next section.

2 Isoperimetric Inequality on Manifolds

2.1 A Domain in the Whole Space

Isoperimetric inequality can be also generalized to domains in manifolds. For 2-manifolds, or, surfaces, Bernstein[9] first found that[14] if C is a convex curve on a sphere of radius R, then

$$L^2 - 4\pi A + \frac{A^2}{R^2} \geqslant (2Rg(R))^2(2\pi + g(R)^2), g(R) = \sin\left(\frac{d}{4(1+2\pi)R}\right),$$

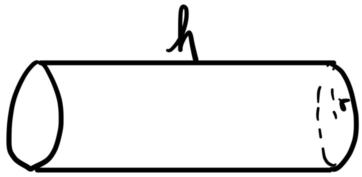
where L is the length of C, A is the area of convex domain bounded by C and d is the minimum width of circular annuli on the sphere containing the given curve.

[9]Felix Bernstein, 1878.1-1956.12, German mathematician, known for proving the Schröder-Bernstein theorem(if $card(A) \leqslant card(B)$ and $card(B) \leqslant card(A)$, then $card(A) = card(B)$).

However, the isoperimetric inequality cannot always take the form $L^2 \geqslant 4\pi A$ on a general surface.

First, since a circle on a sphere satisfies $L^2 = 4\pi A - A^2/R^2$, one should not expect $L^2 \geqslant 4\pi A$ holds for all curves on a sphere. In fact, Beckenbach[10] and Radó[11] showed that if $L^2 - 4\pi A \geqslant 0$ holds for all simply connected domains on a surface, then the Gauss curvature of this surface must be non-positive[15]. So there should be some relationship between the isoperimetric inequality on a surface with its curvature.

(a) A circle on a sphere.



(b) A cylinder.

图 1.3: Two counterexamples where $L^2 \geqslant 4\pi$ A does not holds.

On the other hand, consider a cylinder(where Gauss curvature K=0) with radius r and height h. Its boundary circles have total length $4\pi r$ and its area is $2\pi r h$. Thus the area can be made arbitrary large by increasing h, while the total boundary length remains fixed. As a result, if we drop the simple connectivity, then $L^2 \geqslant 4\pi A$ does not hold for general domains on surfaces even if $K \leqslant 0$.

Now let us state the final version of the isoperimetric inequality on surfaces. It was found by Ionin in 1969[16].

Theorem 2.1. *Let D be a domain on a complete surface. Suppose D has area A, Euler characteristic χ, and length of boundary L. For any real number λ, let $\omega_\lambda^+ = \int_D (K-\lambda)_+$, where K is the Gauss curvature. Then*

$$L^2 \geqslant 2(2\pi\chi - \omega_\lambda^+)A - \lambda A^2.$$

Next, let us turn to higher dimensional cases. This time, we have the famous Lévy[12]-Gromov[13] inequality, which was proved by Lévy for convex hypersurfaces in Euclidean space in 1922[17] and later by Gromov for general cases in 1980[18]..

Theorem 2.2. *Suppose M is a n-dimensional closed manifold, $D \subset M$ is a compact domain with smooth boundary ∂D. Suppose $\mathrm{Ric}_M \geqslant K > 0$, where K is a constant. Let S be a n-dimensional sphere with constant Ricci curvature K, and B be a spherical cap in S such that*

$$\frac{|D|}{|M|} = \frac{|B|}{|S|}.$$

Then

$$\frac{|\partial D|}{|M|} \geqslant \frac{|\partial B|}{|S|}.$$

[10]Edwin Ford Beckenbach, 1906.7-1982.9, American mathematician, led the development of the graduate program in mathematics in UCLA.

[11]Tibor Radó, 1895.6-1965.12, Hungarian mathematician.

[12]Paul Lévy, 1886.9-1971.12, French mathematician, introduced some fundamental concepts in probability theory.

[13]Mikhael Gromov, 1943.12-, Russian-French mathematician, had many revolutionary contributions to geometry.

If we take $M = S$ and let $K \to 0$, then the inequality goes back to the usual isoperimetric inequality in Euclidean spaces.

Theorem 2.2 is a corollary of Lévy-Heintze-Karcher comparison theorem, which bounds the Jacobian determinant of the exponential map in a manifold with sectional curvature bounded below by that in a constant sectional curvature manifold[19]. Based on optimal transport and needle decompositions, Klartag gave an alternative approach to Theorem 2.2[20]. This method was generalized to metric measure spaces by Cavalletti and Mondino in 2017. They showed that if (X, d) is a metric space with Borel probability measure m and, roughly speaking, X has n-dimensional Ricci curvature bounded below by $K > 0$, then for every Borel set $E \subset X$, we have

$$m^+(E) \geqslant \frac{|\partial B|}{|S|}, \tag{2.1}$$

where B, S is same as that in Theorem 2.2.

$$m^+(E) := \liminf_{\epsilon \to 0^+} \frac{m(E^\varepsilon) - m(E)}{\varepsilon}$$

denotes the Minkowski content of E and $E^\varepsilon = \{x \in X : \exists y \in E, d(x, y) < \varepsilon\}$ is the ε-neighborhood of E. If X is the manifold in Theorem 2.2 and m is the normalized volume measure of M, then (2.1) goes back to Lévy-Gromov inequality.

2.2 Minimal Submanifolds

There are various studies of isoperimetric inequality on minimal submanifolds as well. Recall that if we denote the standard connection on $\mathbb{R}^n$ by $\bar{D}$, then the second fundamental form of a submanifold $\Sigma \subset \mathbb{R}^n$ is defined as $\mathrm{I\!I}(X, Y) = (\bar{D}_X Y)^\perp$, where X, Y are tangent vector fields to Σ. The mean curvature vector of Σ, denoted by H, is defined as the trace of $\mathrm{I\!I}$. Σ is called minimal if and only if $H = 0$ on Σ.

In 1921, Carleman[14] proved that every two-dimensional simply-connected minimal surface in $\mathbb{R}^n$ satisfies a sharp[15] isoperimetric inequality $L^2 \geqslant 4\pi A$[22]. Carleman's result is a natural consequence by observing the following fact. Denotes the position vector in $\mathbb{R}^n$ by x. If Σ is an arbitrary submanifold in $\mathbb{R}^n$, then

$$\Delta_\Sigma x = \Delta_\Sigma(x_1, \ldots, x_n) = H(x), \forall x \in \Sigma.$$

This means that if Σ is minimal, then the coordinate functions on Σ are harmonic. Moreover, if Σ is simply connected, then these coordinate functions can be defined on the unit disk in $\mathbb{R}^2 = \mathbb{C}$. As a result, we can use the technique from complex analysis to get the desired estimate.

But this result was not satisfying. A simple closed curve in $\mathbb{R}^3$ may bound a minimal surface with higher genus. Besides, a minimal surface can have disconnected boundary. On the other hand, although we showed that $L^2 \geqslant 4\pi A$ did not hold for all surfaces, we still believe it should hold for all minimal surfaces. This is because an argument shows that $L^2 \geqslant 4\pi A$ is valid for all area-minimizing submanifolds and the area-minimizing property inspires the definition of the minimal submanifold[28].

After Carleman, many attempts were given to weaken the topological assumption in Carleman's result. For example, a sharp isoperimetric inequality for minimal surfaces with

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[14] Torsten Caleman, 1892.7-1949.1, Swedish mathematician.

[15] A sharp inequality means that we can determine when will the equality hold, and non-sharp means that the equality never appears.

connected boundary[23] and for doubly-connected minimal surfaces[24]. Here we give a brief proof of the case where the boundary is connected. This proof was given by Li[16], Schoen[17] and Yau[18] in 1984[25].

Theorem 2.3. *Suppose Σ is a minimal surface in $\mathbb{R}^n$ with boundary connected. Let L be the length of $\partial\Sigma$, A be the area of Σ, then*

$$L^2 \geqslant 4\pi A.$$

Proof. Let $r(x) = |x|$, then

$$\Delta_\Sigma r^2 = \sum_{i=1}^n \Delta_\Sigma x_i^2 = 2\sum_{i=1}^n \left(x_i \Delta_\Sigma x_i + |\nabla^\Sigma x_i|^2\right) = 4.$$

Translating S suitably, we may assume $\int_{\partial\Sigma} x_i = 0, \forall i$.By $\nabla^\Sigma r = x^T/r$ and Wirtinger's inequality, we have

$$4A = 2\int_{\partial\Sigma} r\frac{\partial r}{\partial\eta} \leqslant 2\int_{\partial\Sigma} r \leqslant 2L^{\frac{1}{2}}\left(\int_{\partial\Sigma}\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \leqslant \frac{1}{\pi}L^{\frac{3}{2}}\left[\int_{\partial\Sigma}\sum_{i=1}^n\left(\frac{dx_i}{ds}\right)^2\right]^{\frac{1}{2}} = \frac{1}{\pi}L^2,$$

where η denotes the outer co-normal to $\partial\Sigma$ and s is the arc length parameter of $\partial\Sigma$. □

With some extra arguments, the above proof can in fact derive the equality for those surfaces with two boundary components.

There were also some non-sharp isoperimetric inequalities established for all minimal surfaces. Leon Simon showed that $L^2 \geqslant 2\pi A$ holds for all minimal surfaces. This result was never published. Simon's argument follows the proof of Theorem 2.3. Set $r(x, y) = |x - y|, x, y \in \mathbb{R}^n$. Again we have $\Delta_\Sigma^x r^2 = 4$ and $\nabla_x^\Sigma r = (x-y)^T/r$, where $(x-y)^T$ is the tangent part of $x - y$ at point x. As a result, we get

$$\Delta_\Sigma^y \log r = \frac{2}{r^2}(1 - |\nabla_y^\Sigma r|^2) \geqslant 2\pi\delta_x, \Delta_\Sigma^x r = \frac{1}{r}(2 - |\nabla_x^\Sigma r|^2) \geqslant \frac{1}{r}.$$

Integrating x, y over Σ respectively, we have

$$2\pi \leqslant \int_\Sigma \Delta_\Sigma^y \log r dy \leqslant \int_{\partial\Sigma}\frac{1}{r}\frac{\partial r}{\partial\eta(y)}d\sigma(y) \leqslant \int_{\partial\Sigma}\frac{d\sigma(y)}{r}, \forall x \in \Sigma,$$

and

$$\int_\Sigma \frac{1}{r}dx \leqslant \int_\Sigma \Delta_\Sigma^x r dx = \int_{\partial\Sigma}\frac{\partial r}{\partial\eta(x)}d\sigma(x).$$

Hence integrating the first inequality for x over Σ yields

$$2\pi A \leqslant \int_\Sigma\int_{\partial\Sigma}\frac{1}{r}d\sigma(y)dx = \int_{\partial\Sigma}\int_\Sigma \frac{1}{r}dxd\sigma(y) \leqslant \int_{\partial\Sigma}\int_{\partial\Sigma}\frac{\partial r}{\partial\eta(x)}d\sigma(x)d\sigma(y) \leqslant L^2.$$

Stone improved Simon's proof in 2003[26]. By exchanging x, y in

$$2\pi A \leqslant \int_{\partial\Sigma}\int_{\partial\Sigma}\partial r/\partial\eta(x)d\sigma(x)d\sigma(y),$$

then adding up, he derived $L^2 \geqslant 2\sqrt{2}A$.

[16] Peter Li, 1952.4-, American mathematician.

[17] Richard Schoen, 1950.10-, American mathematician, known for the resolution of the Yamabe problem.

[18] Everybody knows who he is.

In higher dimensions, we have the famous Michael-Simon Sobolev inequality[27]. This inequality is of the type

$$\left(\int_\Sigma f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leqslant C_n \int_\Sigma (|\nabla^\Sigma f| + f|H|), \tag{2.2}$$

where Σ is a n-dimensional submainfold in $\mathbb{R}^{n+1}$ and f is a compactly supported positive function on Σ. Taking $f = 1$, we get a non-sharp isoperimetric inequality of minimal hypersurfaces in $\mathbb{R}^{n+1}$. In 2010, Castillon improved the constant in (2.2) via optimal transport[29].

Recently, Brendle generalized (2.2) into arbitrary codimension settings.

Theorem 2.4. *Suppose Σ is a compact n-dimensional submanifold of $\mathbb{R}^{n+m}$ with boundary $\partial\Sigma$, $m \geqslant 2$. Let f be a positive smooth function on Σ, then*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2|H|^2} + \int_{\partial\Sigma} f \geqslant n\left(\frac{(n+m)|B^{n+m}|}{m|B^m|}\right)^{\frac{1}{n}} \left(\int_\Sigma f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}. \tag{2.3}$$

Moreover, if $m = 2$ and the equality holds, then f is constant and Σ is a flat round ball.

Since we have the induction formula $(n+2)|B^{n+2}| = 2|B^2||B^n|$, (2.3) implies a sharp isoperimetric inequality for compact n-dimensional submanifolds in $\mathbb{R}^{n+2}$:

$$|\partial\Sigma| \geqslant n|B^n|^{\frac{1}{n}}|\Sigma|^{\frac{n-1}{n}}.$$

We can always embed $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$, so it also provides a sharp isoperimetric inequality in codimension 1.

Let us sketch the main ideas in the proof of Theorem 2.4. Brendle was inspired by Osserman's work, say, Theorem 1.2.

Sketch of proof. We only consider the case where Σ is connected since the disconnected one can be easily derived from the connected one. (2.3) is homogeneous. Scaling, we can assume

$$\int_\Sigma \sqrt{f^2|H|^2 + |\nabla^\Sigma f|^2} + \int_{\partial\Sigma} f = n\int_\Sigma f^{\frac{n}{n-1}}.$$

This normalization ensures the existence of the solution to the Neumann problem

$$\begin{cases} \operatorname{div}_\Sigma(f\nabla^\Sigma u) = nf^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2|H|^2} & \text{in} \quad \Sigma, \\ \langle \nabla^\Sigma u, \eta\rangle = 1 & \text{on} \quad \partial\Sigma. \end{cases}$$

Consider

$$\begin{aligned} \Omega &:= \{x \in \Sigma \setminus \partial\Sigma : |\nabla^\Sigma u| < 1\}, \\ U &:= \{(x,y) : x \in \Sigma \setminus \partial\Sigma, y \in T_x^\perp\Sigma, |\nabla^\Sigma u|^2 + |y|^2 < 1\}, \\ A &:= \{(x,y) \in U : D_\Sigma^2 u(x) - \langle \mathrm{II}(x), y\rangle \geqslant 0\}, \end{aligned}$$

and a map

$$\Phi : U \to \mathbb{R}^{n+m}, \Phi(x,y) = \nabla^\Sigma u(x) + y, \forall (x,y) \in U.$$

First follow the Theorem 1.2, we have $\Phi(A) = B^{n+m}$. A calculation shows the Jacobian determinant of Φ is

$$\det D\Phi(x,y) = \det(D_\Sigma^2 u(x) - \langle \mathrm{II}(x), y\rangle), \forall (x,y) \in U.$$

Then, using the mean value inequality, Cauchy's inequality and the equation of u, we get

$$0 \leqslant \det D\Phi(x,y) \leqslant f(x)^{\frac{n}{n-1}}, \forall (x,y) \in A.$$

We now apply the change of variables formula to the map Φ. For all $0 \leqslant \sigma < 1$,

$$\begin{aligned}
&|B^{n+m}|(1-\sigma^{n+m}) \\
=&\int_{\{\xi \in \mathbb{R}^{n+m}:\sigma^2<|\xi|^2<1\}} 1 d\xi \\
\leqslant&\int_{\Omega}\left(\int_{\{y\in T_x^{\perp}\Sigma:\sigma^2<|\Phi(x,y)|^2<1\}} |\det D\Phi(x,y)| 1_A(x,y) dy\right) d\mathrm{vol}(x) \\
\leqslant&\int_{\Omega}\left(\int_{\{y\in T_x^{\perp}\Sigma:\sigma^2<|\nabla^{\Sigma}u(x)|^2+|y|^2<1\}} f(x)^{\frac{n}{n-1}} dy\right) d\mathrm{vol}(x) \\
=&|B^m|\int_{\Omega}\left[(1-|\nabla^{\Sigma}u(x)|^2)^{\frac{m}{2}} - (\sigma^2-|\nabla^{\Sigma}u(x)|^2)_+^{\frac{m}{2}} f(x)^{\frac{n}{n-1}}\right].
\end{aligned}$$

Since $m \geqslant 2$, we have $b^{\frac{m}{2}} - a^{\frac{m}{2}} \leqslant \frac{m}{2}(b-a)$ for all $0 \leqslant a \leqslant b \leqslant 1$. Hence

$$(1-|\nabla^{\Sigma}u(x)|^2)^{\frac{m}{2}} - (\sigma^2-|\nabla^{\Sigma}u(x)|^2)_+^{\frac{m}{2}} \leqslant \frac{m}{2}\left[(1-|\nabla^{\Sigma}u(x)|^2) - (\sigma^2-|\nabla^{\Sigma}u(x)|^2)_+\right] \leqslant \frac{m}{2}(1-\sigma^2).$$

Putting these facts together, we obtain $|B^{n+m}|(1-\sigma^{n+m}) \leqslant \frac{m}{2}|B^m|(1-\sigma^2)\int_{\Omega} f^{\frac{n}{n-1}}, \forall 0 \leqslant \sigma < 1$. Then divide by $1-\sigma$ and take the limit as $\sigma \to 1^-$. This gives

$$(n+m)|B^{n+m}| \leqslant m|B^m|\int_{\Omega} f^{\frac{n}{n-1}} \leqslant m|B^m|\int_{\Sigma} f^{\frac{n}{n-1}}.$$

Thus, we conclude that

$$\int_{\Sigma}\sqrt{f^2|H|^2+|\nabla^{\Sigma}f|^2} + \int_{\partial\Sigma} f = n\int_{\Sigma} f^{\frac{n}{n-1}} \geqslant n\left(\frac{(n+m)|B^{n+m}|}{m|B^m|}\right)^{\frac{1}{n}}\left(\int_{\Sigma} f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

□

A year later, by the same idea with much more complicated arguments, Brendle derived some same type inequalities for manifolds with non-negative curvature. He showed that if M is a n-dimensional complete non-compact manifold with non-negative Ricci curvature, D is a compact domain in M and f is a positive smooth function on D, then

$$\int_D |\nabla^{\Sigma} f| + \int_{\partial D} f \geqslant n|B^n|^{\frac{1}{n}}\theta^{\frac{1}{n}}\left(\int_D f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}},$$

where θ is the asymptotic volume ratio of M,

$$\theta := \lim_{r\to\infty}\frac{|\{p\in M : d(p,q)<r\}|}{|B^n|r^n} \leqslant 1.$$

The equality holds if and only if M is isometric to Euclidean space, D is a ball, and f is constant. Besides, suppose the dimension of M is $n+m$ and M has non-negative sectional curvature, Σ is a compact n-dimensional submanifold of M with boundary Σ and f is a positive smooth function on Σ. If $m \geqslant 2$, then

$$\int_{\Sigma}\sqrt{|\nabla^{\Sigma}f|^2+f^2|H|^2} + \int_{\partial\Sigma} f \geqslant n\left(\frac{(n+m)|B^{n+m}|}{m|B^m|}\right)^{\frac{1}{n}}\theta^{\frac{1}{n}}\left(\int_{\Sigma} f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

If $m=2$, the equality holds if and only if M is isometric to Euclidean space, Σ is a flat round ball, and f is a constant. These two inequalities provides sharp isoperimetric inequalities for manifolds with non-negative curvature.

3 Further Discussion

As we have seen, there are numerous results on isoperimetric inequality. However, there are still many related topics on isoperimetric inequality, which are impossible to be included in this short article. For example, the sharp isoperimetric inequality in $\mathbb{R}^n$ is equivalent to the optimal constant in Sobolev's inequality. Also, isoperimetric inequality is related to the first eigenvalue of Laplacian. There are many other methods which can be applied to isoperimetric inequality not mentioned in the previous section. Besides, isoperimetric inequality has many interesting applications such as the concentration inequality in probability theory. For these topics, one may refer to [1] or [2]. For more details, readers can refer to [3].

4 Acknowledgment

The author highly appreciates prof. Wei Yong. Under his suggestion, the author began to read some advanced papers on geometry, including [1], [30] and [31]. The author received excellent scientific research training from him.

This article is based on two speeches given in Hua Loo-Keng seminar, section 1 for the first speech(12, 2021) and section 2 for the second speech(3, 2022). The author should also express his gratefulness to the classmates who attended the seminar. They carefully listened to the author and pointed out a mistake the author made in the seminar. The author got better understanding of the contents from their discussion.

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RSA 及素性检测简介

李书亮

摘 要

在现代通讯及计算机技术出现之前, 数论基本上只是数学家的 “游戏”, 但后来数论 (包括现代代数学) 在通讯安全领域出现了无穷的妙用. 本文将列出数论的一些结论, 并给出 RSA 算法的介绍和经典的 Miller-Rabin 素性测试的描述和正确性证明.

1 预备知识与记号

设 B 为一个集合, 用 $\#B$ 表示其中元素的个数. 文中的 p, q 均表示素数, $\varphi(n)$ 为欧拉函数, 即 $[1, n]$ 中与 n 互素的整数的个数. 一个数 n 可以被分解成 $2^k m$, 其中 m 为奇数, 方便起见我们称 2^k 为 n 的 even part, m 为 n 的 odd part. $(\mathbb{Z}/n\mathbb{Z})^*$ 表示所有与 n 互素的数在乘法和同余意义下构成的群.

定理 1.1. *设 $n = \prod_{p|n} p^{a_p}$ 为 n 的素因子分解, 则 $\varphi(n) = \prod_{p|n}(p-1)p^{a_p - 1}$.*

定理 1.2. *$(\mathbb{Z}/n\mathbb{Z})^*$ 为循环群, 当且仅当 $n = 2, 4, p^e, 2p^e$, 其中 p 是奇素数, e 是任意正整数.*

定理 1.3. *在一个 n 阶循环群中, $x^e = 1$ 的解的个数为 $\gcd(n, e)$.*

定理 1.4 (中国剩余定理)**.** *设 $n = \prod_{p_i|n} p^{a_{p_i}}$ 为标准分解, 则同余方程组 $M \equiv b_i \bmod p_i^{a_{p_i}}$ 对一组固定的 (b_i), $0 \le b_i < p_i^{a_{p_i}}$ 有且仅有一个满足 $0 \le M < n$ 的 M 成立.*

推论 1.5. *$\#\{M^d \equiv r \bmod n\} = \prod_{p|n} \#\{M^d \equiv r \bmod p^{a_p}\}$.*

定理 1.6 (Fermat)**.** *若 p 为素数, 则对于 $x \nmid p$, 有 $x^{p-1} \equiv 1 \bmod p$.*

定理 1.7 (Euler)**.** *若 $x \in (\mathbb{Z}/n\mathbb{Z})^*$, 则 $x^{\varphi(n)} \equiv 1 \bmod n$.*

2 RSA 简介

先简要介绍对称加密和非对称加密. 加密是指从明文到密文的一个映射 (明文和密文一般可以认为是二进制串或者一个数字), 也许会有一个参数, 称为密钥, 即密文 = C(明文, 加密密钥), 解密是指从密文到明文的映射, 明文 = D(密文, 解密密钥). 根据加密密钥和解密密钥是否相同, 我们把加密算法分为对称加密和非对称加密.

如果只有公开信道 (即有被窃听的可能), 对称加密的一个重要问题在于通信双方如何共同确定一个密钥 (称为密钥分发问题). 私下见面商量这种朴素的办法在互联网时代效率较低, 所以我们需要其他的方式. 密钥分发有两个比较经典的办法: Merkle's Puzzles [3] 和 Diffie-Hellman 密钥交换 [4]. 下面简要介绍后者 (是一个简单的数论知识的运用):

(设通信双方分别是 Alice 和 Bob)

- Alice 和 Bob 先公开选择一个素数 p, 和一个对应的原根 g.
- Alice 和 Bob 各自分别选择一个秘密整数 a 和 b.
- Alice 计算 $A \equiv g^a \bmod p$ 发给 Bob, Bob 计算 $B \equiv g^b \bmod p$ 发给 Alice.
- 根据 $(g^a)^b \equiv (g^b)^a \bmod p$, Alice 和 Bob 就有了一个共同的数字作为密钥.

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稿件信息: 3 月 31 日来稿, 4 月 23 日第一次修改, 4 月 25 日接受. 稿件编辑: 包恺成.

上述的算法的安全性基于给定 g^a 和 g^b 来算出 g^{ab} 是很难的 (在模 p 意义下), 这被称为离散对数问题. p 选为素数的一个考虑是 A, B 分别有 $p-1$ 种可能 (因为 $g^x \bmod p$ 对 $1 \leqslant x \leqslant p-1$ 均不同), 减少了被破解的可能性.

关于非对称加密, RSA[1]是世界上第一个做到加密密钥和解密密钥不同的现代加密算法. 我们先介绍 RSA 的实现:

(1) 选取两个不同的素数 p, q.

(2) 计算 $n = pq, \varphi(n) = (p-1)(q-1)$.

(3) 选取与 $\varphi(n)$ 互素的奇数 e, 并算出 e 关于模 $\varphi(n)$ 的逆元 d.

(4) $P = (e, n)$ 公开, 作为公钥, $S = (d, n)$ 保密, 作为私钥.

在通信中, 我们认为信息就是一个数字 $M (M < n)$, 加密算法:

$$P(M) = M^e \bmod n.$$

其中 $P(M)$ 称为 M 对应的密文, 给定一个密文 C, 解密算法:

$$S(C) = C^d \bmod n.$$

其中 $S(C)$ 即为对应的原文. 关于如何快速计算一个数关于模另一个数的逆元的算法, 请参考扩展欧几里得算法.

定理 2.1. *RSA 算法是正确的, 即* $(M^d)^e \equiv M^{(ed)} \equiv M \bmod n$.

证明. 若 M 与 p 互素 (即 p 不是 M 的因子), 由于 $ed = 1 + k(p-1)(q-1)$, 由费马小定理, $M^{ed} \equiv M(M^{p-1})^{k(q-1)} \equiv M \bmod p$. 若 M 不与 p 互素, 则 $M^{ed} \equiv 0 \equiv M \bmod p$. 同理有 $M^{ed} \equiv M \bmod q$, 由中国剩余定理, $M^{ed} \equiv M \bmod n$. □

RSA 的安全性由对大数进行因子分解的困难性保证.

以下是几个有意思的事情: 给定 (e, n) 和 (d, n), 可以有效的把 n 进行素因数分解 (在此有效是指在一个关于 n 的二进制表示位数的一个多项式的时间内能够完成, 在数论算法里有效一般就是指关于位数的多项式时间内能完成); 若 $d < n^{1/4}/3$, 则对给定的 e, 可以有效得到 d;

电子签名: $M^d \bmod n$ 称为私钥持有者对 M 的签名, 公众可以用公钥对 $M^d \bmod n$ 实施加密算法来验证该签名 (注意到公钥和私钥的地位其实是等价的);

blinding 攻击: 攻击者想要获取 M 的签名, 那么可以随机选取一个 $r \in (\mathbb{Z}/n\mathbb{Z})^*$, 令 $M' = r^e M \bmod n$, 则 M' 的签名与 M 的签名一致.

以上几个攻击的证明和更多有趣的对于 RSA 的攻击可以参考 [1].

3 素性测试

由 RSA 的算法可以看出, 想要在实际中运用 RSA 算法 (也包括其他的很多算法) 必须要能找到比较大的素数, 如果算法中的素数选的比较小, 那么即使是暴力破解也能破解私钥, 因此如何快速找到大素数成为人们关心的问题.

数论中有一个著名的素数定理:

定理 3.1. *设* $\pi(n)$ *为从* 1 *到* n *里所有素数的个数, 则* $\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = 1$.

[1]Rivest-Shamir-Adleman, 1977. 但据英国政府称, 一位英国数学家 Clifford Cocks 在 1973 年就发明了该算法, 但由于其在政府通信部门工作, 该结果不能公开发表.

上述的比值在 n 不是很大的时候与 1 也非常接近, 这意味着, 如果我们想找一个二进制表示有 500 位左右的素数, 则只要随机寻找大约 $300 \sim 400$ 个数即可, 如果只寻找奇数, 还能进一步减少这个数字, 那么接下来的问题就转化为如果快速判断一个数是否是素数, 这个问题称为素性测试.

最简单最原始的办法就是试除法: 如果要判断 n 是否是素数, 只要逐个去除 2 到 $\sqrt{n}$ 内的所以数即可, 如果有一次试除余数为 0, 则 n 就不是素数, 否则 n 就是素数.

但问题在于, 这个算法对于位数比较大的数来进行素性测试, 是不现实的, 因为如果一个二进制数有 k 位, 则试除的最坏可能要进行 $2^{k/2}$ 的量级次, 如果 k 为 500, 这个次数就和宇宙中的原子数目差不多.

下一种测试办法的想法来自于费马小定理, 如果 n 为素数, 那么对任何一个小于 n 的数 m 都有 $m^{n-1} \equiv 1 \bmod n$, 如果有一个数不满足这个等式, 我们就可以自信的认为 n 不是素数, 这个办法成为费马测试. 但很遗憾就算对所有的 m 都满足这个等式, 也不能说明 n 就是素数, 这样的 n 被称为 Carmicheal 数, 最小的 Carmicheal 数是 561.

虽然费马测试有着难以克服的缺点, 但对其稍做改进 (也许是大改进) 就有了著名的 Miller-Rabin 素性测试算法, 首先给出一个引理:

引理 3.2. *设 p 是一个素数, $x^2 \equiv 1 \bmod p$ 在模 p 意义下只有两个解:1 和 -1.*

证明. 若 $x^2 \equiv 1 \bmod p$, 则 $x^2 - 1 = (x-1)(x+1)$ 是 p 的倍数, 则必有 $x+1$ 与 $x-1$ 中有一个数 p 的倍数, 即 $x \equiv 1 \bmod p$ 与 $x \equiv -1 \bmod p$ 必有一个成立. □

对于一个素数 p, $p - 1 = 2^l k$ 其中 k 是一个奇数, 给定一个小于 p 的数 m, 首先有 $m^{p-1} \equiv 1 \bmod p$, 则 $m^{2^{l-1}k} \equiv 1 \bmod p$ 或 $m^{2^{l-1}k} \equiv -1 \bmod p$ 有一个成立, 若 $m^{2^{l-1}k} \equiv 1 \bmod p$ 成立, 则有 $m^{2^{l-2}k} \equiv 1 \bmod p$ 或 $m^{2^{l-2}k} \equiv -1 \bmod p$ 有一个成立, 如此继续, 直至有一个 $i, m^{2^i k} \equiv -1 \bmod p$ 成立, 或者 $m^k \equiv 1 \bmod p$, 这便是 Miller-Rabin 算法的核心思想.

Algorithm 1: Miller-Rabin test

```
Data: 一个奇数 n, 一个小于 n 的正整数 m
把 n − 1 分解成 2^l k, 其中 k 为奇数
计算 m^(2^l k) 模 n 的余数, 如果不是 1, 则返回 false
for i ← l − 1 to 0 do
    计算 m^(2^i k) 模 n 的余数, 如果不是 1 或 n − 1, 返回 false
    如果是 n − 1 结束循环
end
返回 true
```

如果 (n, m) 没有通过上述算法, 则必有 n 是一个合数, 称 m 为 n 是合数的一个证据. 用费马测试束手无策的 $n = 561$ 为例, 取 $m = 2$, $561 - 1 = 35 \times 2^4, 2^{560} \equiv 1 \bmod 561, 2^{280} \equiv 1 \bmod 561, 2^{140} \equiv 140 \bmod 561$, 从而 561 不是素数. 算法竞赛里的一个有用的事实: 只需要使用前十二个素数 $(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37)$ 进行 Miller-Rabin 测试, 就可以判断 2^{64} 以内的所有数是否是素数.

定理 3.3. *设 n 为一个大于 9 的奇合数, $n = 2^l k$, k 是奇数, 记*

$$B = \left\{ x \in (\mathbb{Z}/n\mathbb{Z})^* : x^k \equiv 1 \bmod n \ or \ x^{k2^i} \equiv -1 \bmod n \ for\ some\ 0 \le i < l \right\}$$

我们有

$$\frac{\#B}{\varphi(n)} \le \frac{1}{4}$$

注 3.4. B 中的 x 即为能通过 *Miller-Rabin* 测试的 (n,x).

证明. 先设 $n=\prod_{p|n}p^{a_p}$, 设 r 为满足 $2^r|(p-1)$ 对每一个 n 的素因子 p 都成立的最大的 r. 记

$$C=\{x\in(\mathbb{Z}/n\mathbb{Z})^*:x^{k2^{r-1}}\equiv\pm1\bmod n\}$$

我们有 $B\subseteq C$, 事实上, 若 $x\in B$, 若 $x^k\equiv1\bmod n$, 则显然 $x\in C$, 否则设 $x^{k2^i}\equiv-1\bmod n$, 对每一个 n 的素因子 p, 都有 $x^{k2^i}\equiv-1\bmod p$, 则 $x^{k2^{i+1}}\equiv1\bmod p$, 设 x 在 $(\mathbb{Z}/p\mathbb{Z})^*$ 中的阶为 d, 则 2^{i+1} 是 d 的因子 ($k2^{i+1}$ 是 d 的倍数, 但 $k2^i$ 不是 d 的倍数), 又有 $d|(p-1)$, 从而 $2^{i+1}|(p-1)$, 即 $i+1\le r$, 即得 $B\subseteq C$.

使用中国剩余定理便有, $x^{k2^{r-1}}\equiv1\bmod n$ 的解的个数是, 对每一个 n 的素因子 p, $x^{k2^{r-1}}\equiv1\bmod p^{a_p}$ 的解的个数的乘积, 又由于 $(\mathbb{Z}/p^{a_p}\mathbb{Z})^*$ 是循环群, 则解的个数为 $\gcd((p-1)p^{a_p-1},k2^{r-1})=\gcd(p-1,k)2^{r-1}$(因为 p 整除 n, 从而不能整除 $n-1$, 不能整除 k), 因此有

$$\#\{x\in(\mathbb{Z}/n\mathbb{Z})^*:x^{k2^{r-1}}\equiv1\bmod n\}=\prod_{p|n}\gcd(p-1,k)2^{r-1}$$

若 $x^{k2^{r-1}}\equiv-1\bmod n$, 则 $x^{k2^{r-1}}\equiv-1\bmod p^{a_p}$, 对每一个 n 的素因子 p 成立, 由 $x^{k2^{r-1}}\equiv-1\bmod p^{a_p}$ 和 $x^{k2^{r-1}}\equiv1\bmod p^{a_p}$ 的解集的并集为 $x^{k2^r}\equiv1\bmod p^{a_p}$ 的解集, 根据上述讨论后者的解集大小为 $\prod_{p|n}\gcd(p-1,k)2^r$, 从而 $x^{k2^{r-1}}\equiv-1\bmod p^{a_p}$ 解集大小也为 $\prod_{p|n}\gcd(p-1,k)2^{r-1}$. 从而

$$\#\{x\in(\mathbb{Z}/n\mathbb{Z})^*:x^{k2^{r-1}}\equiv-1\bmod n\}=\prod_{p|n}\gcd(p-1,k)2^{r-1}$$

因此易有

$$\#C=2\prod_{p|n}\gcd(p-1,k)2^{r-1}$$

$$\frac{\#C}{\varphi(n)}=2\prod_{p|n}\frac{\gcd(p-1,k)2^{r-1}}{(p-1)p^{a_p-1}}$$

先假设若有 $\frac{\#B}{\varphi(n)}$ 超过了 $1/4$. 因此

$$\frac{1}{4}<2\prod_{p|n}\frac{\gcd(p-1,k)2^{r-1}}{(p-1)p^{a_p-1}}\tag{*}$$

因为 $\gcd(p-1,k)2^{r-1}$ 整除 $(p-1)/2$, 因此右式至多为 2^{1-t}, t 是 n 的素因子的个数, 因此 $t\le2$. 若 $t=2$, 再若有 $a_p\ge2$ 对某个 p 成立, 则右式至多为 $1/(2*3)=1/6$(因为 p 至少为 3). 因此 $n=pq$, 这时有

$$\frac{p-1}{2^r\gcd(p-1,k)}\cdot\frac{q-1}{2^r\gcd(q-1,k)}<2$$

再由 $\gcd(p-1,k)2^r$ 整除 $(p-1)$(p 可以换成 q), 左式的两个分式都是整数, 因此都是 1, 即 $p-1=\gcd(p-1,k)2^r$, $q-1=\gcd(q-1,k)2^r$, 因为 k 为奇数, 所以 $p-1$ 和 $q-1$ 的 odd part 都整除 k, 且 $p-1$ 和 $q-1$ 的 even part 都是 2^r. 因为 $(p-1)(q-1)+(p-1)+(q-1)=pq-1=2^lk$, 因此我们有 $p-1$ 的 odd part 整除 $q-1$, $q-1$ 的 odd part 整除 $p-1$, 从而 $p=q$, 矛盾. 因此只能有 $t=1$, 即 $n=p^a, a\ge2$, 由(*)有 $p^{a-1}<4$, 因此 $p=3,a=2$, 这与 $n>9$ 矛盾. 证毕. ◻

因此, 对于一个很大的奇数 n, 如果随机选取 $m < n$ 进行 Miller-Rabin 测试, 如果进行了 k 次都通过了, 那么可以认为只有 $1/4^k$ 的可能 n 是合数. 这样的算法是非常有趣的, 因为它虽然不保证百分百正确, 但可以让出错的几率任意小.

有趣的是, 这个算法和广义黎曼猜想竟然扯上了关系:

定理 3.5. *如果广义黎曼猜想成立, 则若 n 是奇合数, 则在 1 到 $2(\log n)^2$ 之间必然有一个数作为 n 是合数的证据.*

证明. 参考 [2]. □

该定理说明, 如果广义黎曼猜想成立, 那么 Miller-Rabin 测试其实是一个多项式时间的素性判断算法. 事实上, 世界上第一个理论为证明是多项式时间的确定性素性判定算法是在 2002 年由三位印度数学家提出来的 (Agrawal-Kayal-Saxena primality test).

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A 附录: 快速计算 $a^b \bmod n$

设 b 有二进制表示 $b_n b_{n-1} \dots b_1$.

Algorithm 2: Binary Exponentiation

```
let C_1 = a mod n, R_0 = 1
for i ← 1 to n do
    if b_i = 1 then
        R_i = R_{i-1} * C_i mod n
    else
        R_i = R_{i-1}
    end
    C_{i+1} = C_i * C_i mod n
end
return R_n
```

注. 对于大整数的运算, 所用时间和参与运算的数的二进制表示长度有关, 朴素的算法下, 加法和减法用时和二进制串长度是线性关系, 乘除和取模则是平方关系, 都是多项式时间.

环带多胞体及其关联定向拟阵

曾相如

摘 要

在百花齐放的多面体几何学中, 环带多面体因其独特的组合学性质而被广泛研究. 本文将介绍几个典型的环带多面体, 并介绍环带多面体的基本组合性质. 随后, 我们将视角转向其高维推广: 环带多胞体, 并引入一个重要的组合学工具: 定向拟阵. 我们将借助定向拟阵的语言刻画环带多胞体的组合结构, 并给出其高维体积的具体表达.

1 绪论

众所周知[1], 我们有五种柏拉图立体, 即凸正多面体: 正四面体, 正六面体, 正八面体, 正十二面体以及正二十面体. 对于一个柏拉图立体, 如果我们将每条棱换为与其垂直的另一条线, 使得这条线与所有棱的中点的外接球相切, 我们便能得到其对偶多面体 (图 1.1). 其中, 正四面体和自己互为对偶, 正六面体和正八面体互为对偶, 正十二面体和二十面体互为对偶.

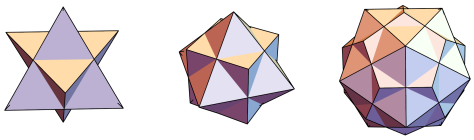

图 1.1: 三对对偶柏拉图立体

熟悉阿基米德立体的朋友这时就知道, 在上述构造下, 正六面体和正八面体的交集是截半立方体, 正十二面体和正二十面体的交集是截半二十面体 (图 1.2).

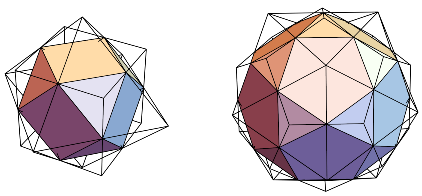

图 1.2: 截半立方体与截半二十面体

如果我们继续考虑上述构造下对偶立体的并的凸包, 我们就能得到两个卡塔兰立体 (Catalan solid, 表示阿基米德立体的对偶立体): 菱形十二面体和菱形三十面体, 它们分别是之前两种多面体的对偶 (图 1.3).

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稿件信息: 3 月 24 日来稿, 4 月 27 日第一次修改, 5 月 4 日第二次修改, 5 月 4 日接受. 稿件编辑: 罗迟.
[1] 更多多面体知识可以参考 [2, §§ 1-2].

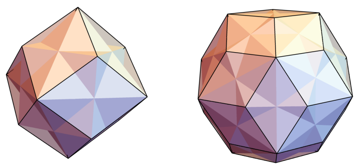

图 1.3: 菱形十二面体与菱形三十面体

其中, 菱形十二面体的每个面都是对角线长度之比为 $1:\sqrt{2}$ 的菱形, 而我们用四个该菱形对应的钝角菱面体恰好可以组合出一个菱形十二面体 (图 1.4). (菱面体即为六个面都是相同的菱形的平行六面体, 在此基础上一个钝角菱面体有两个顶点由三个钝角组成).

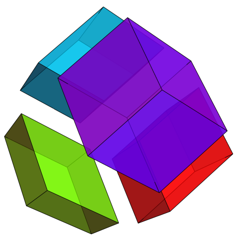

图 1.4: 菱形十二面体的分解

类似的, 菱形三十面体的每个面的对角线长度之比恰好为黄金比例 $1:\frac{1+\sqrt{5}}{2}$, 而我们可以通过二十个该菱形对应的菱面体组合出一个菱形三十面体, 其中钝角菱面体和锐角菱面体各十个 (图 1.5).

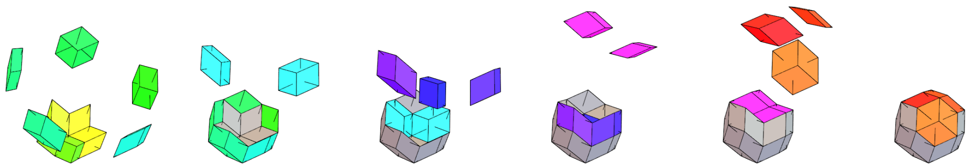

图 1.5: 菱形三十面体的分解

若我们选取菱形十二面体的一条棱, 然后观察所有和该棱平行的棱, 能够发现这些棱以及它们之间的菱形组成了一个**环带 (zone)**, 这个环带由六个菱形组成, 而菱形十二面体上一共有四个这样的环带; 对于菱形三十面体, 每个环带由十个菱形组成, 多面体上一共有六个这样的环带 (图 1.6).

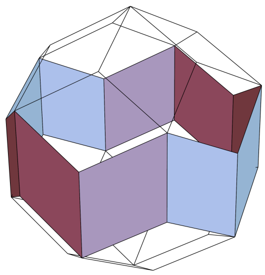

图 1.6: 菱形三十面体上的一个环带

这两个多面体便属于我们接下来要介绍的**环带多面体 (zonohedron)**. 虽然这两个多面体的分解看似很特殊, 但我们之后会证明, 一般的环带多面体上也会有类似这两种多面体的分解.

2 环带多面体

我们先考虑一般的**由平行四边形围成的凸多面体**. 因为所有面都是平行四边形, 和之前一样, 每一条棱决定了一个**环带**, 在这个环带上的每个面都有两个和给定棱平行且相等的边. 同时, 每一个面都恰好属于两个不同的环带, 而这两个环带除了共享这个面, 同时由于凸多面体表面的单连通性, 它们还会共享另一个面, 则这两个面全等且对应边平行. 由于多面体的凸性, 这样的面恰有两个. 这两个面可以组成一个平行六面体, 我们视其体对角线的两个端点为对应点, 从而这四对对应点连线共享中点. 通过考虑与这一对面相邻的其他相互对应的面, 可以发现这个对应点关系可以不依赖于面的选择, 同时所有对应点连线共享中点. 换句话说, 这个多面体有**中心对称性**.

接下来我们探究一下与由平行四边形围成的凸多面体相关的数量关系. 设这个多面体有 n 种不同方向的棱, 那么这个多面体上恰好有 n 个不同的环带. 对于菱形十二面体和菱形三十面体, 这个数字分别是 4 和 6. 由于每个环带都和其他 $n-1$ 个环带有两个公共面, 一个环带总共就有 $2(n-1)$ 个面, 从而这个多面体总共有 $n(n-1)$ 个不同的面. 此外, 由于每个方向的边都有 $2(n-1)$ 条, 一共有 $2n(n-1)$ 条边. 由欧拉公式, 我们得到了这个多面体的顶点数: $n(n-1)+2$. 读者可以自行验证一下之前的两个例子.

我们称空间中一族向量 $V=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)\in\mathbb{R}^{3\cdot n}$ 为一个**向量配置 (vector configuration)**, 这些向量可以重复, V 可以视为一个多重集. 如果 V 中任意两个向量不共线, 我们称之为**简单**的; 如果任意三个向量不共面, 则我们称之为**处于一般位置**的, 此时一个向量配置可以视为向量的集合. 对于任意一个集合 $A\subseteq\{1,\dots,n\}$ 我们定义 ${}_{-A}V=(u_1,\dots,u_n)$, 其中

$$u_i=\begin{cases} v_i, & \text{若} i\notin A \\ -v_i, & \text{若} i\in A \end{cases}.$$

我们称 ${}_{-A}V$ 为 V 的一个**重定向**. 重定向诱导了向量配置间的一个等价关系. 现在我们可以说, 在相差一个重定向的意义下, 一个由平行四边形围成的多面体唯一确定了一个处于一般位置的向量配置 (视为集合), 这个配置中的每一个向量等于多面体中的 $2(n-1)$ 个相互平行的边.

反过来, 这一族相差一个重定向的向量配置也唯一决定着这个多面体. 我们选取这个多面体上的一个点为原点, 并考虑过原点的一个多面体的支撑面. 我们可以给向量配置重定向,

使得每个向量都朝向支撑面的内侧. 接下来, 我们我们考虑点集

$$\left\{\sum_{i=1}^{n}\lambda_i\boldsymbol{v}_i : 0 \le \lambda_i \le 1\right\} \subseteq \mathbb{R}^3,$$

即这些向量表示的线段的闵可夫斯基和, 它是空间中的凸集. 注意到多面体的每一个顶点都可以沿着棱连接到原点, 而按照定义每条棱可以用某个 $\boldsymbol{v}_i$ 表示, 从而顶点坐标都形如

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_n\boldsymbol{v}_n,\ \lambda_i \in \{0,1\}.$$

于是多面体的顶点都在该凸集内, 从而整个多面体都包含在该凸集内. 另一方面, 我们可以对 n 归纳证明该凸集也包含在该多面体里面. 当 $n \le 3$ 时, 结论显然成立; 假设对 $n-1$ 成立, 但存在 $\{\lambda_i\} \in [0,1]^n$ 使得 $\sum_{i=1}^{n}\lambda_i\boldsymbol{v}_i$ 不在多面体中. 那么, 我们可以取 $1 \le m \le n$ 使得 $\sum_{i=1}^{m-1}\lambda_i\boldsymbol{v}_i$ 在多面体中, 而 $\sum_{i=1}^{m}\lambda_i\boldsymbol{v}_i$ 不在多面体中. 考虑去除 $\boldsymbol{v}_m$ 对应的环带, 则多面体其余的面组成了两个连通分支, 恰有一个包含原点. 将不包含原点的连通分支整体平移 $-\boldsymbol{v}_m$, 其和包含原点的分支组成了一个新的多面体, 恰好是一个由平行四边形围成的凸多面体, 有 $n-1$ 种不同方向的棱 (图 1.7).

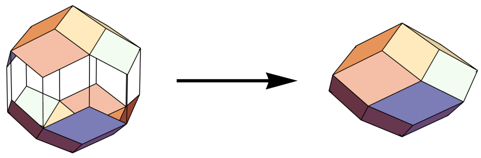

图 1.7: 去除菱形三十面体的环带

但是, 由凸性容易知道 $\sum_{i=1}^{m-1}\lambda_i\boldsymbol{v}_i$ 并不会出现在这个新的多面体中, 和归纳假设矛盾. 现在我们证明了这个凸集和多面体重合, 而注意到对向量配置重定向相当于对这个凸集进行平移, 不会改变形状和方向, 于是我们可以说这个多面体也由向量配置唯一确定.

接下来我们从向量配置的角度来研究多面体上的顶点. 我们还是和之前一样选取一个处于一般位置的向量配置. 注意到一个凸集的顶点即为其在某一个方向上的唯一最远点. 我们过原点作出垂直于该方向的平面, 那么向量配置中和该方向落在平面同一侧的所有向量之和一定是多面体中该方向的最远点. 同时由于最远点是唯一的, 平面上不能包含向量配置中的任何向量, 且平面另一侧的向量之和恰好是反方向的最远点, 这两点是对应点. 于是, 多面体中对应点对的数量和我们用一个过原点的平面将向量配置一分为二的方案数一致. 我们接着考虑单位球面, 每一个向量的法平面都对应着球面上的一个大圆, 这些大圆无三圆共点, 它们将球面分割成若干对对径的区域. 同时, 每个过原点的平面的法线都对应着球面上一对对径点. 此时问题转化为求解球面上有多少对对径的区域. 通过简单的归纳, 我们知道这个数目为 $\frac{n(n-1)}{2}+1$. 每一对区域对应着向量配置的一个分割, 从而对应着多面体的一对对应点, 于是我们又一次得到了多面体的顶点数量 $n(n-1)+2$. 此外, 类似分析可知, 多面体的每一对对应面对应着一个过原点的平面, 其恰好包含配置中的两个向量, 从而我们得到多面体的面数 $2\binom{n}{2} = n(n-1)$. 从上述分析我们也可以看出来, 一个处于一般位置的向量配置一定能给出一个由平行四边形围成的凸多面体.

注. *图 1.8 所示的五个大圆由菱形三十面体构造而来. 可见, 这种构造实际上给出了球面上的"对偶多面体".*

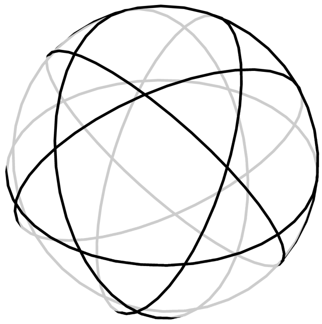

图 1.8: 向量配置给出的一族大圆

考虑平面中 n 个不共线向量构成的简单向量配置, 它给出了一个中心对称的 $2n$ 边形. 那么对于一般的空间中的简单向量配置, 它也可以对应一个中心对称的凸多面体, 但每个面不一定是平行四边形. 这个配置可能有若干个子集由共面向量组成, 这些共面向量组成了一个平面向量配置, 从而在多面体上表现为一对中心对称的多面体. 我们称这样的多面体为**环带多面体**. 除了之前的两种卡塔兰多面体之外, 截角八面体, 大斜方截半立方体以及大斜方截半二十面体都是环带多面体 (图 1.9).

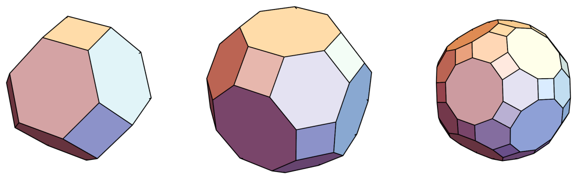

图 1.9: 阿基米德立体中的三种环带多面体, 从左到右依次为: 截角八面体, 大斜方截半立方体, 大斜方截半二十面体

对于平面中的简单向量配置 $V=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$, 选取该平面的法向量 $\boldsymbol{u}$ 和恰当的 $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ 使得 $\widehat{V}=(\boldsymbol{v}_1+\lambda_1\boldsymbol{u},\boldsymbol{v}_2+\lambda_2\boldsymbol{u},\dots,\boldsymbol{v}_n+\lambda_n\boldsymbol{u},\boldsymbol{u})$ 是空间中处于一般位置的向量配置. 那么, $\widehat{V}$ 能张成一个由平行四边形围成的多面体, 而且其沿 $\boldsymbol{u}$ 的投影恰好是由 V 张成的 $2n$ 边形. 我们去掉 $\boldsymbol{u}$ 对应的环带, 则剩下的两个连通分支的投影分别给出了 $2n$ 边形的一个镶嵌 (图 1.10).

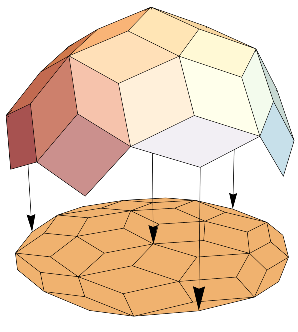

图 1.10: 投影给出的镶嵌

这两个镶嵌相互对称, 且镶嵌中的平行四边形可以和 V 中的向量对一一对应, 于是我们可以借此算出 $2n$ 边形的面积.

那么, 对于空间中的向量配置我们是否也可以这样操作呢? 是否一个环带多面体可以表示为高维物体的投影呢? 答案是肯定的. 接下来我们将一步到位, 直接讨论任意维空间的情形.

3 环带多胞体及其关联定向拟阵

为了方便叙述, 下面我们先规定一些记号

- $\{+,-,0\} \coloneqq \{+1,-1,0\}$, 其中的元素称为**符号**;
- $\operatorname{sign}(x) \coloneqq \begin{cases} 0, & 若x=0 \\ \dfrac{x}{|x|}, & 若x\neq 0 \end{cases} \in \{+,-,0\}$, 表示实数 x 的符号;
- $[n] \coloneqq \{1,2,\ldots,n\}$;
- $\{+,-,0\}^E \coloneqq \{X\colon E\to\{+,-,0\}\}$, 其中的元素称为集合 E 上的**符号向量**.

对于符号向量 $X,Y\in\{+,-,0\}^E$,

- $X_e \coloneqq X(e)$, 对于 $e\in E$. 当 $E=[n]$ 时, 我们也会把 X 表示成向量 $(X_1,X_2,\ldots,X_n)$;
- $X^+ \coloneqq \{e\in E : X_e=+\}, X^- \coloneqq \{e\in E : X_e=-\}, X^0 \coloneqq \{e\in E : X_e=0\}$, 我们也会把 X 表示成有序对 (X^+,X^-);
- $-X \coloneqq (X^-,X^+)$;
- $\underline{X} \coloneqq X^+\cup X^- = E\setminus X^0$, 称为 X 的**支集**;
- $\mathbf{0} \coloneqq (\emptyset,\emptyset)$, 即零向量;
- $X\leq Y$, 若有 $X^+\subseteq Y^+$ 且 $X^-\subseteq Y^-$, 这在 $\{+,-,0\}^E$ 上定义了一个偏序;
- $S(X,Y) \coloneqq \{e\in E : X_e=-Y_e\neq 0\}$, 称为 X 和 Y 的**分离集**;
- $X\circ Y \coloneqq (X^+\cup(Y^+\setminus X^-), X^-\cup(Y^-\setminus X^+))$, 或者 $(X\circ Y)_i \coloneqq \begin{cases} X_i, & 若X_i\neq 0 \\ Y_i, & 若X_i=0 \end{cases}$, 称为 X 和 Y 的**复合**.

对于向量 $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$, 以及 (凸) 集合 $K,L\subseteq\mathbb{R}^d$,

- $[\boldsymbol{a},\boldsymbol{b}] \coloneqq \{\lambda\boldsymbol{a}+(1-\lambda)\boldsymbol{b} : \lambda\in[0,1]\}$, 为两个向量间的线段;
- $K+L \coloneqq \{\boldsymbol{u}+\boldsymbol{v} : \boldsymbol{u}\in K,\boldsymbol{v}\in L\}$, 为两个集合间的**闵可夫斯基和**;
- $H_{\boldsymbol{a}} \coloneqq \left\{\boldsymbol{v}\in\mathbb{R}^d : \boldsymbol{a}\cdot\boldsymbol{v}=0\right\}, H_{\boldsymbol{a}}^+ \coloneqq \left\{\boldsymbol{v}\in\mathbb{R}^d : \boldsymbol{a}\cdot\boldsymbol{v}>0\right\}, H_{\boldsymbol{a}}^- \coloneqq \left\{\boldsymbol{v}\in\mathbb{R}^d : \boldsymbol{a}\cdot\boldsymbol{v}<0\right\}$;
- $F_{\boldsymbol{a}}(A) \coloneqq \{\boldsymbol{v}\in K : \boldsymbol{a}\cdot\boldsymbol{v}\geq\boldsymbol{a}\cdot\boldsymbol{u},\forall\boldsymbol{u}\in K\}$. 当 $\boldsymbol{a}\neq\mathbf{0}$ 时, 该式表示 K 在 $\boldsymbol{a}$ 方向上的最远点集;
- $\mathcal{F}(K) \coloneqq \{\emptyset,K\}\cup\left\{F_{\boldsymbol{a}} : \boldsymbol{a}\in\left(\mathbb{R}^d\right)^*\right\}$. 当 K 为闭的凸集时, $\mathcal{F}(K)$ 中的元素称为 K 的**面**[2], 其中不为 K 或 $\emptyset$ 的面称为**真面**, 极大的真面称为**超面 (facet)**.

下面我们给出向量配置和环带多面体的高维推广:

定义 3.1. d 维空间中的**向量配置**是一列向量 $V=(\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_n)\in\mathbb{R}^d$, 不同排序下的 V 视为是同构的. 如果无任意两个向量共线, 我们称其为**简单**的. 如果任意 d 个向量间线性无关, 我们称其为**处于一般位置**的. 向量配置的**重定向**和三维的情形一致.

定义 3.2. 向量配置 V 给出的**环带多胞体** ***(zonotopal)*** $Z(V)$ 定义为闵可夫斯基和

$$Z(V) \coloneqq \sum_{i=1}^{n}[-\boldsymbol{v}_i,\boldsymbol{v}_i]\subseteq\mathbb{R}^d,$$

其为关于原点中心对称的闭凸集.

[2]此处定义参考了 [3, § 2.4], 里面有更加严谨的叙述.

可见, 空间中的环带多胞体就是环带多面体, 而平面中的环带多胞体就是中心对称的凸 $2n$ 边形. 研究环带多胞体的组合性质往往化归为研究其向量配置上的线性相关性, 为此我们引入一个工具, 它把向量配置上的线性相关性抽象为一族符号向量:

定义 3.3. V 的**定向拟阵** *(oriented matroid)* 由一族**余向量** *(covector)* 给出:

$$\mathcal{L}(V) := \left\{\sigma(\boldsymbol{c}) : \boldsymbol{c} \in \mathbb{R}^d\right\} \subseteq \{+,-,0\}^{[n]},$$

其中 $\sigma : \mathbb{R}^d \to \{+,-,0\}^{[n]}$ 定义为

$$\begin{aligned}\sigma(c) &:= \left(sign(\boldsymbol{c}\cdot\boldsymbol{v}_1), sign(\boldsymbol{c}\cdot\boldsymbol{v}_2), \ldots, sign(\boldsymbol{c}\cdot\boldsymbol{v}_n)\right)\\ &= \left(\left\{e\in[n] : \boldsymbol{v}_e \in H_{\boldsymbol{c}}^+\right\}, \left\{e\in[n] : \boldsymbol{v}_e \in H_{\boldsymbol{c}}^-\right\}\right).\end{aligned}$$

我们可以认为 $\mathbf{0} \neq X = \sigma(\boldsymbol{a}) \in \mathcal{L}(V)$ 对应着 d 维空间中的一些过原点的定向超平面 $H_{\boldsymbol{a}}$, X^+ 和 X^- 分别表示超平面正侧 $H_{\boldsymbol{a}}^+$ 和负侧 $H_{\boldsymbol{a}}^-$ 中的向量, 而 X^0 表示着超平面包含的向量. 进一步我们可以发现, 若 $Y = \sigma(\boldsymbol{b})$, 则分离集 $S(X,Y)$ 代表着 $\left(H_{\boldsymbol{a}}^+ \cap H_{\boldsymbol{b}}^-\right) \cup \left(H_{\boldsymbol{a}}^- \cap H_{\boldsymbol{b}}^+\right)$ 中的向量. 对于一般的定向拟阵, 我们可以用四条关于余向量的公理来定义:

定义 3.4. 我们称一族符号向量 $\mathcal{L} \subseteq \{+,-,0\}^E$ 为一个**定向拟阵**的全体**余向量**, 当且仅当满足如下所有条件:

(L0) $\mathbf{0} \in \mathcal{L}$,

(L1) $X \in \mathcal{L} \Rightarrow -X \in \mathcal{L}$,

(L2) $X, Y \in \mathcal{L} \Rightarrow X \circ Y \in \mathcal{L}$,

(L3) 若 $X, Y \in \mathcal{L}$ 且 $e \in S(X,Y)$, 则存在 $Z \in \mathcal{L}$ 使得 $Z_e = 0$, 且对任意 $f \notin S(X,Y)$, 有 $Z_f = (X \circ Y)_f = (Y \circ X)_f$.

其中的 E 称为该定向拟阵的**基础集**.

对一个向量配置的定向拟阵而言, (L0) 和 (L1) 是显然成立的. 对于 (L2), 设 $X = \sigma(\boldsymbol{a}), Y = \sigma(\boldsymbol{b})$, 我们可以使 $\boldsymbol{a}$ 向 $\boldsymbol{b}$ 旋转一个小角度, 则 $H_{\boldsymbol{a}}$ 同时也向 $H_{\boldsymbol{b}}$ 旋转一个小角度, 所得的超平面能给出 $X \circ Y$. 如果 $e \in S(X,Y)$, 我们让 $\boldsymbol{a}$ 向 $\boldsymbol{b}$ 旋转至 $\boldsymbol{c}$, 使其与 $\boldsymbol{v}_e$ 正交, 则一定有 $H_{\boldsymbol{c}}^+ \supseteq H_{\boldsymbol{a}}^+ \cap H_{\boldsymbol{b}}^+, H_{\boldsymbol{c}}^- \supseteq H_{\boldsymbol{a}}^- \cap H_{\boldsymbol{b}}^-$, 取 $Z = \sigma(\boldsymbol{c})$ 便能满足 (L3). 于是, 向量配置的定向拟阵符合一般的定向拟阵的定义. 一个非零余向量如果拥有极小的支集, 我们便称其为**余环路 (cocircuit)**. 定向拟阵 $\mathcal{L}$ 的全体余环路构成的集合记为 $\mathcal{C}^* = \mathcal{C}^*(\mathcal{L})$.

接下来我们给出余向量对于环带多面体的几何意义. 给定向量配置 $V \subseteq \mathbb{R}^{d\cdot n}$, 环带多面体 $Z := Z(V)$ 以及定向拟阵 $\mathcal{L} := \mathcal{L}(V)$. 对于每个余向量 $X \in \mathcal{L}$, 我们定义

$$Z_X := \sum_{i\in X^0} [-\boldsymbol{v}_i, +\boldsymbol{v}_i] + \sum_{i\in X^+} \boldsymbol{v}_i - \sum_{i\in X^-} \boldsymbol{v}_i.$$

那么 $Z_X \subseteq Z$. 选取 $\boldsymbol{a} \in \mathbb{R}^d$ 使得 $X = \sigma(\boldsymbol{a})$, 注意到 $FD_{\boldsymbol{a}}(K+L) = F_{\boldsymbol{a}}(K) + F_{\boldsymbol{a}}(L)$ 对任意凸集 K, L 成立, 我们有

$$\begin{aligned}Z_X &= \sum_{i\in H_{\boldsymbol{a}}} [-\boldsymbol{v}_i, +\boldsymbol{v}_i] + \sum_{i\in H_{\boldsymbol{a}}^+} \boldsymbol{v}_i - \sum_{i\in H_{\boldsymbol{a}}^-} \boldsymbol{v}_i\\ &= \sum_{i\in H_{\boldsymbol{a}}} F_{\boldsymbol{a}}([-\boldsymbol{v}_i, +\boldsymbol{v}_i]) + \sum_{i\in H_{\boldsymbol{a}}^+} F_{\boldsymbol{a}}([-\boldsymbol{v}_i, +\boldsymbol{v}_i]) + \sum_{i\in H_{\boldsymbol{a}}^-} F_{\boldsymbol{a}}([-\boldsymbol{v}_i, +\boldsymbol{v}_i])\\ &= F_{\boldsymbol{a}}\left(\sum_{i=1}^{n} [-\boldsymbol{v}_i, +\boldsymbol{v}_i]\right)\\ &= F_{\boldsymbol{a}}(Z).\end{aligned}$$

于是, Z_X 恰好是 Z 的一个面 (当 $\boldsymbol{a}=0$ 时, $Z_X=Z$). 反过来, Z 的任意一个非空面一定形如 $F_{\boldsymbol{a}}$, 从而这个面由 $Z_{\sigma(\boldsymbol{a})}$ 给出. 于是, $X\mapsto Z_X$ 给出了全体余向量和 Z 的所有非空面之间的双射[3]. 注意到 Z_X 实际上是由下标属于 X^0 的那些向量给出的一个环带多胞体, 它的维数由这个向量集的秩决定. 特别地, Z_X 是一个超面当且仅当 X 是一个余环路, Z_X 是一个点当且仅当 X 是极大的余向量.

我们接着考虑 V 处于一般位置的情形. 此时, X^0 给出的向量集的秩恰为 $\left|X^0\right|$, 从而 X 对应一个 $\left|X^0\right|$ 维的环带多面体. 下面我们给出具体的面数公式.

定理 3.1. 对于 $0\le k<d$, Z 的 k 维面个数 $f_k^{(n,d)}$ 满足

$$f_k^{(n,d)}=2\binom{n}{k}\sum_{m=0}^{d-k-1}\binom{n-k-1}{m}, \tag{3.1}$$

特别地

$$f_0^{(n,d)}=2\sum_{m=0}^{d-1}\binom{n-1}{m}. \tag{3.2}$$

证明. 我们对 $d\ge 2$ 归纳证明.

当 $d=2$ 时, Z 为 $2n$ 边形, 满足 $f_0^{(n,2)}=2n=2\binom{n}{0}\left(\binom{n-1}{0}+\binom{n-1}{1}\right), f_1^{(n,2)}=2n=2\binom{n}{1}\binom{n-2}{0}$, (3.1) 式成立.

当 $d>2$ 时, 假设对任意 $d'<d, n\ge d', 0\le k<d'$ 有 (3.1) 成立. 对于 $1\le k<d$, 一个 k 维面对应的余向量 $X\in\mathcal{L}$ 满足 $\left|X^0\right|=k$. 给定 k 元集 $I\in[n]$, 设 $U=\operatorname{span}\left(\{\boldsymbol{v}_i:i\in I\}\right)$, 为 I 对应的向量张成的空间. 那么, $X^0=I$, 当且仅当 $X=\sigma(\boldsymbol{a})$, 其中 $\boldsymbol{a}$ 满足 $U\subseteq H_{\boldsymbol{a}}$, 而且对于任意 $j\notin I$, 我们有 $\boldsymbol{v}_j+U\notin H_{\boldsymbol{a}}/U\subseteq\mathbb{R}^d/U$, 也就是说 $d-k$ 维空间 $\mathbb{R}^d/U$ 中的超平面 $H_{\boldsymbol{a}}/U$ 给出了向量配置 $\left(\boldsymbol{v}_j+U\right)_{j\notin I}$ 的一个极大余向量. 由于向量配置 $\left(\boldsymbol{v}_j+U\right)_{j\notin I}$ 也是处于一般位置的, 由归纳假设知极大余向量数量为 $f_0^{(n-k,d-k)}=2\sum\limits_{m=0}^{d-k-1}\binom{n-k-1}{m}$. 由于 k 元子集有 $\binom{n}{k}$ 个, 于是我们得到

$$f_k^{(n,d)}=\binom{n}{k}\cdot 2\sum_{m=0}^{d-k-1}\binom{n-k-1}{m},$$

即为 (3.1). 接下来我们利用欧拉公式

$$\sum_{k=0}^{d-1}(-1)^k f_k^{(n,d)}=\left(1-(-1)^d\right) \tag{3.3}$$

来求解 $f_0^{(n,d)}$. 注意到

$$\begin{aligned}
&\sum_{k=0}^{d-1}(-1)^k\cdot 2\binom{n}{k}\sum_{m=0}^{d-k-1}\binom{n-k-1}{m}\\
=&\sum_{k=0}^{d-1}(-1)^k\binom{n}{k}\sum_{m=0}^{d-k-1}\left(\left(1-(-1)^{d-k-m+1}\right)+\left(1-(-1)^{d-k-m}\right)\right)\binom{n-k-1}{m}\\
=&\sum_{k=0}^{d-1}(-1)^k\binom{n}{k}\sum_{m=0}^{d-k-1}\left(1-(-1)^{d-k-m}\right)\left(\binom{n-k-1}{m}+\binom{n-k-1}{m-1}\right)\\
=&\sum_{k=0}^{d-1}(-1)^k\binom{n}{k}\sum_{m=0}^{d-k-1}\left(1-(-1)^{d-k-m}\right)\binom{n-k}{m}
\end{aligned}$$

[3] 事实上, 如果我们给 $\mathcal{L}$ 添加一个最大元, 这里的双射便给出了 $\mathcal{L}$ 和 Z 的面格 (face lattice) 的反序格同构, 见 [1, § 2.2].

$$
\begin{aligned}
&= \sum_{0\le k+m<d} \left((-1)^k - (-1)^{d-m}\right) \binom{n}{k}\binom{n-k}{m} \\
&= \sum_{0\le k+m<d} \left((-1)^k - (-1)^{d-m}\right) \binom{n}{k+m}\binom{k+m}{m} \\
&= \sum_{l=0}^{d-1} \left((-1)^l - (-1)^d\right) \binom{n}{l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} \\
&= \sum_{l=0}^{d-1} \left((-1)^l - (-1)^d\right) \cdot 0^l \\
&= \left(1 - (-1)^d\right),
\end{aligned}
$$

这与 (3.3) 是一致的, 于是我们得到 $f_0^{(n,d)} = 2\sum\limits_{m=0}^{d-1}\binom{n-1}{m}$. □

注. *除了利用欧拉公式, 我们也可以仿照三维的情形, 将 $f_0^{(n,d)}$ 的求解转化为求解 S^{d-1} 被 n 个处于一般位置的过原点超平面所划分的区域个数, 此时可以对 n 归纳得到 (3.2), 或者可以参考 [9, § 5E].*

4 定向拟阵的收缩以及环带多胞体的体积

在三维的情形中, 我们讨论了两种对于环带多面体的操作: 一种是将一个环带去除, 将剩下的两个连通分支拼接; 另一种是沿着一条棱的方向向二维投影. 现在我们将这两种操作抽象成定向拟阵上的两种构造:

定义 4.1. *对于定向拟阵 $\mathcal{L} \subseteq \{+,-,0\}^{[n]}$, 其在元素 n 上的***删除** *$\mathcal{L}\backslash n$ 和***收缩** *$\mathcal{L}/n$ 分别定义为*

$$\mathcal{L}\backslash n := \left\{X \in \{+,-,0\}^{[n-1]} : \text{存在}\sigma \in \{+,-,0\} \text{ 使得 } (X,\sigma) \in \mathcal{L}\right\}$$

以及

$$\mathcal{L}/n := \left\{X \in \{+,-,0\}^{[n-1]} : (X,0) \in \mathcal{L}\right\}.$$

可以想象, 删除代表着将对应的环带去除, 而收缩代表着沿对应的棱投影. 如果 $\mathcal{L} = \{(X,0) : X \in \mathcal{L}/n\}$, 我们称元素 n 为一个**自环 (loop)**. 在之前的讨论中, 我们将一个平面中的向量配置变为了一个空间中的向量配置, 这对应着定向拟阵的**单元素提升**:

定义 4.2. *设 $\mathcal{L} \subseteq \{+,-,0\}^{[n]}$ 是一个定向拟阵, $g \notin [n]$. 一个定向拟阵 $\widehat{\mathcal{L}} \subseteq \{+,-,0\}^{[n]\cup g}$ 被称为 $\mathcal{L}$ 的一个***单元素提升***, 如果 $\widehat{\mathcal{L}}/g = \mathcal{L}$, 且 g 不是 $\widehat{\mathcal{L}}$ 中的自环. 我们定义*

$$O\left(\widehat{\mathcal{L}}\right) := \left\{X \in \{+,-,0\}^{[n]} : (X,+) \in \widehat{\mathcal{L}}\right\}.$$

在三维情形中, 我们提及了通过投影可以给出平面 $2n$ 边形的一个镶嵌. 现在我们严格定义环带多胞体的镶嵌.

定义 4.3. *设 $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \in \mathbb{R}^{d\cdot n}$ 是一个向量配置. 如果一族符号向量 $O \subseteq \{+,-,0\}^{[n]}$ 满足:*

(i) $\bigcup_{X\in O} Z_X = Z(V)$,

(ii) 若 $X \in O$, 且符号向量 Y 给出了 Z_X 的一个面 Z_Y, 则 $Y \in O$,

(iii) 若 $X, Y \in O$, 则 $Z_X \cap Z_Y$ 同时是 Z_X 与 Z_Y 的面,

*则 O 称为 $Z(V)$ 的一个***环带多胞体镶嵌***.*

可以验证, 沿着一条棱进行投影给出的镶嵌的确是一个环带多胞体镶嵌. 那是否每一个镶嵌都由这样的投影给出呢? 事实证明这是错误的. 但如果我们利用定向拟阵的收缩, 确实可以得到所有的环带多胞体镶嵌. 具体地, 我们有如下的 Bohne-Dress 定理.

定理 4.1 (A. Dress, 见 [6]). *设 $V \in \mathbb{R}^{d \cdot n}$ 是一个向量配置, $Z \coloneqq Z(V)$ 为其给出的环带多胞体, $\mathcal{L} \coloneqq \mathcal{L}(V)$ 是其对应的定向拟阵. 那么, 映射*

$$\left\{\widehat{\mathcal{L}} \subseteq \{+,-,0\}^{[n]\cup g} : \widehat{\mathcal{L}} \text{ 是} \mathcal{L} \text{ 的单元素提升}\right\} \to \left\{\mathcal{O} \subseteq \{+,-,0\}^{[n]} : \mathcal{O} \text{ 是} Z \text{ 的环带多胞体镶嵌}\right\}$$
$$\widehat{\mathcal{L}} \mapsto \mathcal{O}\left(\widehat{\mathcal{L}}\right)$$

是良好定义的双射.

该映射的良定性以及双射性的证明都不平凡, 感兴趣的读者可以参考 [6]. 不过我们有必要回顾一下二维情形时我们的构造. 事实上, 对于一般的 d 维向量配置 $V = (\boldsymbol{v}_i)) \subseteq \mathbb{R}^{n \cdot d}$, 我们都可以构造出一个处于一般位置的向量配置 $\widehat{V} = (\boldsymbol{v}_1 + \lambda_1 \boldsymbol{u}, \boldsymbol{v}_2 + \lambda_2 \boldsymbol{u}, \ldots, \boldsymbol{v}_n + \lambda_n \boldsymbol{u}, \boldsymbol{u})$, 使得 $\widehat{\mathcal{L}} \coloneqq \mathcal{L}\left(\widehat{V}\right)$ 是 $\mathcal{L}(V)$ 的单元素提升, 从而 $\mathcal{O}\left(\widehat{\mathcal{L}}\right)$, 给出了一个 $Z(V)$ 的环带多胞体镶嵌. 特别地, 由于 $\widehat{V}$ 处于一般位置, $\widehat{\mathcal{L}}$ 中每个形如 $(X,+)$ 的余环路一定满足 $\left|X^0\right| = d$, 而反过来任取 $[n]$ 中的 d 元子集, $\widehat{V}$ 中对应向量张成的超平面一定不经过 $\boldsymbol{u}$, 从而能给出形如 $(X,+)$ 的余环路, 从而给出了 $\mathcal{O}\left(\widehat{\mathcal{L}}\right)$ 中的极小元 X. 这样的 X 给出的 Z_X 是一个 (可能退化) 的 d 维平行多胞体 (平行四面体的高维推广), 其超体积可以由 X 对应向量的行列式的绝对值乘以 2^d 给出. 由于 $\mathcal{O}\left(\widehat{\mathcal{L}}\right)$ 给出了 Z 的镶嵌, 而计算镶嵌的超体积时我们只需考虑那些体积非零的 Z_X, 它们一定满足 $\left|X^0\right| = d$. 于是我们得到了如下命题:

命题 4.2 (P. McMullen, 见 [7]).

$$\operatorname{vol}(Z(V)) = 2^d \cdot \sum_{1 \leq i_1 < i_2 < \cdots < i_d \leq n} \left|\det\left(\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \ldots, \boldsymbol{v}_{i_d}\right)\right|.$$

这便完整回答了我们在 § 2 末提出的问题.

5 总结及延伸

我们从两种卡塔兰多面体出发, 介绍了环带多胞体和定向拟阵的概念, 并对于环带多胞体的镶嵌给出了定理 4.1. 除了研究如何将环带多胞体分解成环带多胞体, 我们同时也关心什么样的环带多胞体能够镶嵌整个空间. 事实上, 一个环带多胞体能够通过平移给出空间的镶嵌当且仅当其对应的定向拟阵是**正则**的, 即可以由一个全幺模矩阵的列向量给出. 不使用拟阵语言的原始证明来自 [5], 其拟阵陈述可以参考 [8, Prop. 3.3.4].

平移给出的镶嵌自然具有平移对称性. 但并非所有镶嵌都会具有平移对称性. 事实上, 在 § 1 中我们将一个菱形三十面体分解成两种菱面体, 而我们可以通过这两种菱面体得到一个 "3D 版本的彭罗斯镶嵌", 它具有和菱形三十面体一致的旋转对称性, 但不具有任何平移对称性 (图 1.11). 这是一种**准周期镶嵌 (quasiperiodic tiling)**, 它和准晶体的结构息息相关, 见 [4].

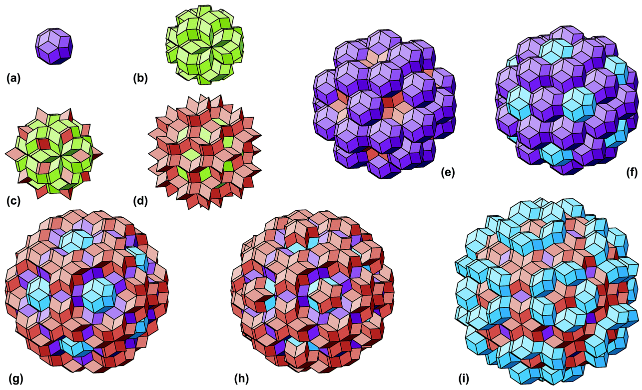

图 1.11: 两种菱面体得到四种基元后给出的准周期镶嵌

定向拟阵的研究在计算几何, 拓扑学, 代数几何等领域都有着联系和应用, 笔者水平有限无法在此列举. 感兴趣的读者可以参考 [1] 进行学习.

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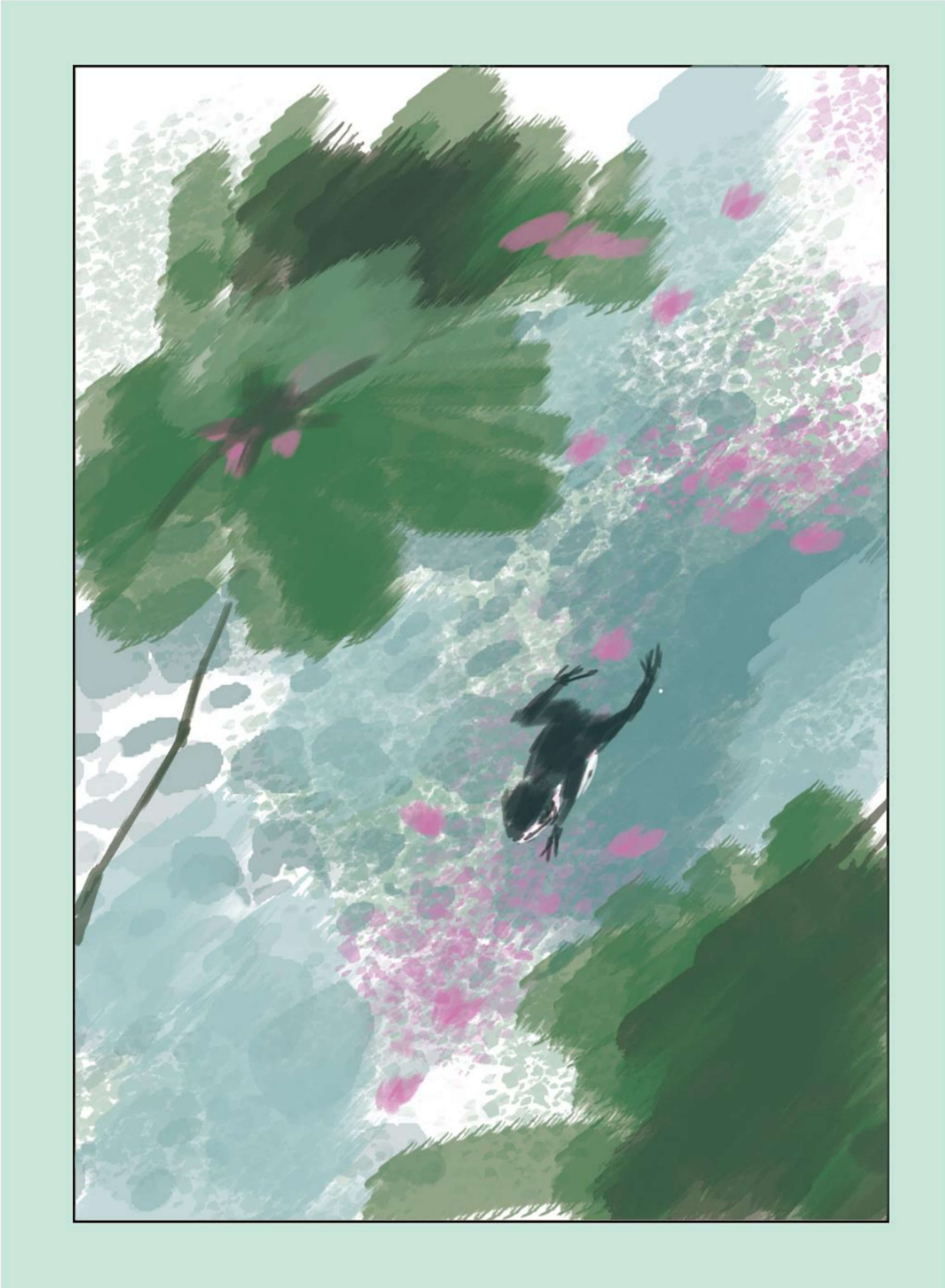
星辰绚烂

缀满夜空

指引心之所向


星辰



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——《西江月 · 夜行黄沙道中》

What is a Motive?

Barry Mazur

我们从代数拓扑的情形出发, 考虑有限维连通单纯复形 X. 现在的问题是从它的一阶上同调中可以获知多少 X 本身的拓扑信息? 或者说, 如果只知道 $\mathrm{H}^1(X,\mathbb{Z})$, 可以在多大程度上还原出 X?

下面的这个答案几乎是一种同义重复 (tautology)[1]. 令 GX 为自由 Abel 群 $\mathrm{H}^1(X,\mathbb{Z})$ 的 Pontjagin 对偶 $\mathrm{Hom}(\mathrm{H}^1(X,Z),\mathbb{R}/\mathbb{Z})$, 这是一个紧连通 Abel 李群[2]. 从而 $\mathrm{H}^1(GX,\mathbb{Z})$ 典范同构于 $\mathrm{H}^1(X,\mathbb{Z}) = \mathrm{Hom}(GX,\mathbb{R}/\mathbb{Z})$, 并且有一个典范的同伦类

$$X \to GX$$

诱导了 H^1 上的恒等映射. 于是问题的答案是: 我们可以读出一切可以从 GX 获知的信息, 然而将无从知晓映射 $X \to GX$ 中丢失的信息. 如果考虑一般情况, 即将 $\mathrm{H}^1(X,\mathbb{Z})$ 换做 $\mathrm{H}^n(X,R)$(R 是系数环), Eilenberg-MacLane 空间的理论则给出了一个类似的答案.

现在我们要问, 如果在结构更为丰富的代数几何世界里考虑这个问题, 是否能期待获得一个与上面相似的理论?

在代数拓扑中, 上同调函子被 Eilenberg-Steenrod 公理完全刻画了. 而对代数几何而言情况则更为复杂. 对于域 k 上的代数簇 X, 我们甚至没有一个很好的 $\mathbb{Z}$ 系数上同调理论——除非有嵌入 $k \to \mathbb{C}$, 在这种情况下我们可以考虑代数簇的 “复点”, 从而划归到代数拓扑中的上同调理论. 然而即使 k 可以嵌入 $\mathbb{C}$, 这样得出的上同调也还依赖于嵌入的选取. 为了寻找合适的上同调, 人们建立了许多独立于前面方法的理论. 其中有一些理论需要特殊的基域 k、系数环, 或者对代数簇有一些限制, 还有一些伴随着额外结构和相互间的比较定理而出现: 例如 Hodge 上同调、代数 de Rham 上同调、晶体 (cristalline) 上同调、平展 (étale) 上同调、ℓ-进上同调 (对于每个素数 ℓ), 等等.

对于光滑射影簇, 是否有一种自然而统一的办法来囊括这些上同调[3] 的所有信息?

如果聚焦于 1 阶上同调, 这个问题看起来相当有希望. 对于域 k 上的光滑射影曲线 C, 它的 Jacobi 簇 $J(C)$ 是一个维数等于 C 的亏格 (genus) 的 Abel 簇. $J(C)$ 在 $\overline{k}$ 上的点构成的群定义为 C 上的 0 次除子群商去有理函数所诱导的除子, 并且我们有自然的函子 $C \to J(C)$, 它从光滑射影曲线的范畴映到 k 上 Abel 簇的范畴, 并且保持了全部的 1 阶上同调信息. 这让人想起之前关于 $X \to GX$ 的构造, 然而这里的 $J(C)$ 有着相较于 GX 更为丰富的结构. 作为推广, Albanese 的优美构造对每个 k 上的代数簇 V 给出 k 上的 Abel 簇 $A(V)$. 自然地, 我们进而希望对于高阶上同调寻找类似的构造, 或者说寻找 Eilenburg-MacLane 空间的代数类比作为 Abel 簇的高维替代. 然而这并非易事.

A. Grothendieck 首先描述了一种方案来一统上述诸多上同调论. 他所寻觅的是某种最典范的 “上同调”, 用以来贯通几何对象和之前所举例的所有具体的上同调理论——正如其名, 它如同诸上同调 “机器” 背后的 “动机”(motive) 或 “母体”. 他这样形容这个概念:

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[1]这大概是说, 这个构造其实并没有给出 ‘新’ 的东西, GX 完全是从 H^1 出发构造的, 因此能给出 H^1 的信息.

[2]对绝大对数足够好的空间, $\mathrm{H}^1(X,\mathbb{Z})$ 是一个有限生成 $\mathbb{Z}$-模, 而其挠部分在取对偶后消失, 于是 GX 实为 $\mathrm{rank}(\mathrm{H}^1(X,\mathbb{Z}))$ 个圆 S^1 的乘积空间.

[3]依照传统, 我们会首先将这些上同调群和 $\mathbb{Q}$ 做张量积来使问题简单一些.

"与代数拓扑中的情况不同, 我们将要面对一大批不同的上同调理论, 这令人感到不快. 你会模糊地感觉到这些不同的理论背后应该是相同的东西, 或者说它们 '给出了相同的结果'. 为了把这不同理论的关联性的直觉具体化, 我建立了所谓的 '原相' 概念, 它对每个代数簇定义. 之所以使用这个词, 是我想表达它是代数簇的不同上同调不变量背后的 '共同动机'(或者 '共同原因'). 实际上, 它将诱导一切可能的上同调不变量."([1])

在 [1] 中, Grothendieck 延续这一想法而描述了一个与音乐的类比. 在这个类比中, 他设想的原相上同调 (motivic cohomology) 恰如一首音乐中的动机, 而每个具体的上同调理论则是由动机演化出的主题材料, 分别在各自的调性和节奏上发展出音乐.

我们来考虑一种公理化代数几何中的上同调理论的方法. 这种万有 (universal) 上同调理论应该被定义为一个反变函子 $V \mapsto H(V)$, 它从域 k 上光滑射影代数簇的范畴映到一个足够好的分次 (graded)Abel 范畴 $\mathcal{H}$. 例如, 我们可能会希望每个对应[4](correspondence)$V \to W$ 给出一个上同调上的反变映射. 而且我们要求范畴 $\mathcal{H}$ 能够满足一些上同调理论的特性, 例如 Künneth 公式和 Poincaré 对偶.

Grothendieck 最初对构建这万有上同调理论的尝试是优雅和直接了当的. 他的工作始于射影代数簇范畴, 对其进行了简练而形式性的调整, 从而设法构造了一个范畴——我们会希望它是 Abel 范畴——具有一切所想要的上同调性质. 这大致分为三个步骤. 首先, 以 $\mathbb{Q}$-对应的等价类代替射影簇之间的态射, 其中这等价关系被选定为一切可以根据上同调理论的公理诱导出良定的上同调同态者中最粗糙的. 其次, 添加所谓的 "投射子"(projector) 使之更接近 Abel 范畴, 例如增加本不存在的核和余核. 此外再作调整, 使之可以建立例如 Künneth 公式的关系. 最后, 令 $\mathcal{H}$ 为前面构造的范畴的反范畴. 于是, 根据这种构造, 光滑射影簇上的一切上同调函子都会经过 $\mathcal{H}$. 这样, 我们就得到了想要的 "原相".

$$
\begin{tikzcd}
\text{Smooth Varieties}/k \arrow[r, "H"] \arrow[dr, bend right, "\text{Cohomology Functor } H^*"'] & \mathcal{H} \arrow[d, dashed, "\exists!"] \\
 & \text{Graded Algebra}/k
\end{tikzcd}
$$

然而这个构造也存在着不少缺陷. 一个重要问题是它不能具体地给出原相范畴. 群星般闪耀在代数几何夜空的诸多猜想都旨在以上同调判据给出某种对应的存在, 或者更一般地说, 给出代数链的存在 (例如复数域上的 Hodge 猜想和有限域上的 Tate 猜想), 而它们都可以由对原相范畴的描述来给出解释. 对于原相这一构想的一切实现——即使是在一些特殊情况下——似乎都能够直接得出这些猜想, 反之也是如此.

因此, 人们的梦想便是给出万有上同调函子

$$V \mapsto H(V) \in \mathcal{H},$$

的足够实用的描述, 并且范畴 $\mathcal{H}$ 得以被具体地理解, 同时它最好能够在 1 维情形与之前用 Jacobi 簇给出的描述相容. 同样重要的是, 正像群表示论中不可约表示所扮演的角色那样, 我们也希望对于原相能有一个这样的结果, 即能够在原相范畴中将 $H(V)$ 表示成不可约对象的直和, 从而代表代数簇的不同 "上同调部分", 并且在对应的作用下稳定, 从而可以单独研究每一部分.

最近, Vladimir Voevodsky 和他的合作者提出了可以作为原相之候选的一个相当有趣的范畴, 它由概形上某种极细的 Grothendieck 拓扑下的层所组成. 直观来看, 这是一种 "柔化

[4]对应意即 $V \times W$ 中的一个代数链 (algebraic cycle), 它可以看作是一个多值映射的图像.

的代数几何”, 从而能够恰当地定义同伦, 并且与具体的代数几何相联系而得到一些极好的结果.

对原相理论完整面貌之求索在诸多数学分支中已然成为强大的动力, 例如复分析、代数几何、自守表示论、L 函数、数论等. 而在即将到来的整个 21 世纪中, 情况将依然保持如此.

译后记

本文是 *Notices of the American Mathematical Society* 杂志中 *‘What is ...’* 这一科普短文栏目 2004 年 10 月的一篇, 由数学家 Barry Mazur 撰写. 译者第一次尝试翻译工作, 并且对 motive 理论也不甚了解, 难免有偏颇缺漏之处, 请各位读者指正. 文中主要概念 motive 暂时还没有统一的中文翻译, 这里根据黎景辉教授在他书中的建议译为 “原相”, 大多数人名则为了避免混淆而直接使用英文.

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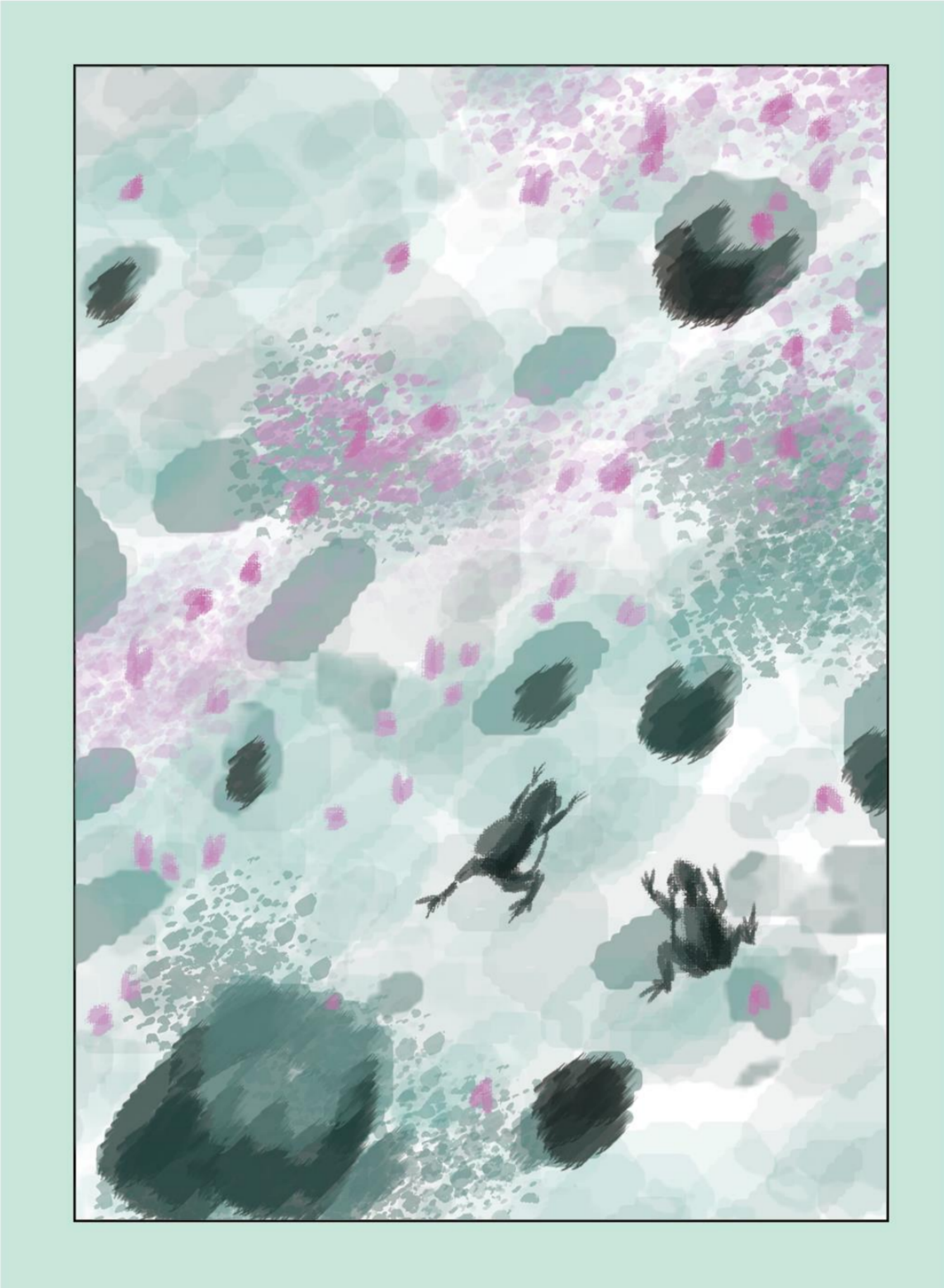

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——《西江月 · 夜行黄沙道中》

流体自由边界问题 (二)

——涡片解的存在性与稳定性

章俊彦

摘 要

考虑两个互不渗透的流体, 如果它们的速度场在交界面的切方向有间断、法方向没有间断, 我们把这个交界面称作涡片 (vortex sheet). 数学上, 二维无旋不可压缩流体的涡片解可以由 Birkhoff-Rott 方程描述, 其解的形式为柯西积分的主值, 从而可以用非卷积奇异积分理论来研究. 对旋度非零的不可压缩流, 当粘性和表面张力都小到可以忽略时, 其涡片解的短时间演化往往都不稳定, 此即流体力学中的 Kelvin-Helmholtz 不稳定性 (KHI). 但表面张力或者一些附加的物理量 (例如磁流体、弹性介质等) 在满足某些特定条件时可以阻止 KHI 的发生. 对可压缩理想流体, 涡片的稳定性还取决于初始速度间断与声速之间的大小关系. 本文将叙述涡片解及其 (不) 稳定性的数学刻画, 介绍若干重要结果, 并展望一些未解决的问题.

1 问题背景与数学刻画

考虑两个互不渗透的不可压缩理想流体 (无粘性、无外力、密度为常数), 它们的运动均由不可压缩欧拉方程表述

$$\begin{cases}(\partial_t + u^\pm \cdot \nabla)u^\pm = -\nabla p^\pm & \text{in } \Omega^\pm(t),\\ \operatorname{div} u^\pm = 0 & \text{in } \Omega^\pm(t).\end{cases} \tag{1.1}$$

其中 $u^\pm, p^\pm$ 分别为流体的速度、压力. 设 $\Gamma(t)$ 为两个流体区域 $\Omega^\pm(t)$ 的交界面, 若在交界面上流体的速度场存在切方向的间断, 则称该交界面为**涡片** (vortex sheet). 此时, 若不考虑表面张力, 则在交界面上形成压力平衡、互不渗透两个现象, 即

$$u^+ \cdot n = u^- \cdot n = \Gamma(t)\text{的运动速度}. \quad (\text{动力学边界条件}) \tag{1.2}$$

$$[p] := p^+ - p^- = 0. \quad (\text{压力平衡条件}) \tag{1.3}$$

其中 n 为 $\Gamma(t)$ 关于 $\Omega^+(t)$ 一侧的单位外法向量. 这样在数学上, 涡片的运动就为两个互不渗透流体的自由交界面问题所刻画. 本文提到的所谓**涡片的稳定性, 是指**(1.1)-(1.3)**这个带有自由界面的初边值问题的 (局部) 适定性.**

若流体的粘性小到可以忽略不计 (例如本文均考虑无粘流体), 给予一个合适的初始扰动, 交界面可能会出现瞬间的崩塌, 这在流体力学里面被称为**开尔文-亥姆霍兹不稳定性** (Kelvin-Helmholtz instability, 简称 KHI). 这种现象往往在行星的大气层中出现, 例如云的形成、木星的大红斑、土星的云带等等. 数学上, 这体现为方程组(1.1)-(1.3)的初边值问题在某类函数空间中不具有局部适定性. 事实上, 早在 80 年代末, 在能量空间中 (Sobolev 空间) 的谬定性 (ill-posedness) 证明就由 [5] 给出. 此外, 在 C^α 型的空间中, 这个初边值问题也是谬定的. 其原因是, 我们可以选取合适的初值, 对该初值进行小的扰动, 演化出的解不满

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稿件信息: 3 月 13 日来稿, 4 月 5 日第一次修改, 4 月 5 日接受. 稿件编辑: 王子达.

足对初值的连续依赖性. 即便如此, 我们仍然可以通过进一步放松函数空间的要求, 或者考虑其它物理量对自由界面的稳定作用, 来设法证明涡片问题的适定性, 并对某些特解进行更深入的研究. 本文将主要介绍四种情况:

1. 二维无旋流: 此时流体的运动完全由自由界面的运动所决定, 而后者可以用复分析的方法化为 Birkhoff-Rott 方程, 其解的形态可以用调和分析中的奇异积分理论进行研究.
2. 表面张力: 表面张力对流体交界面的稳定作用可以抵消上述的不稳定性, 从而阻止交界面的崩塌. 具体而言, 就是把边界条件(1.3)换成

$$[p] = \sigma\mathcal{H}, \tag{1.4}$$

其中常数 $\sigma > 0$ 为给定的**表面张力系数**, $\mathcal{H}$ 为自由界面 $\Gamma(t)$ 的平均曲率. 当表面张力非零时, 我们在 Sobolev 空间 (能量空间) 的框架下至少可以证明局部适定性.
3. 其它物理量带来的稳定性作用: 例如磁流体的电流-涡片问题可以由类似的 MHD 方程自由界面问题来刻画, 若两个磁流体中的磁场在交界面上不共线, 则会在自由界面形成磁剪切层, 即使在没有表面张力的情况下我们也可以证明其局部适定性. 此外, 若考虑弹性介质, 则弹性形变在满足某些特殊条件的时候也能形成对涡片的稳定性作用.
4. 若假设流体可压缩, 则密度不再是常数, 流体中将出现声波的传播. 研究表明, 可压缩流的涡片稳定性除了与上述提到的三个因素有关之外, 还与流体中的声速有关. 当可压缩流的初始速度满足 "超音速条件" 时, 即使没有表面张力、没有其它物理量的参与, 我们仍能在二维的情况下得到局部适定性的结果. 而考虑其它物理量时, 在亚音速域的稳定性条件就比不可压的情况复杂得多. 可压缩无粘流体的涡片稳定性问题, 其数学研究大多仍停留在线性稳定性的层面. 其非线性稳定性, 以及表面张力、外界物理量、声速三者之间的相互影响, 以及与不可压缩流 KHI 的关系, 这些问题的数学研究至今仍在进行中.

2 二维无旋流: Birkhoff-Rott 方程解的存在性与实解析性

考虑一个更特殊的情况: 二维无旋流. 此时, 自由界面的运动完全决定了流体的运动. 这是因为无旋假设 $\nabla\times u = 0$ 可以使得我们将流体的速度场写成向量势的梯度 $u = \nabla\varphi$ 的形式, 再将其代入 $\nabla\cdot u = 0$ 即得 $\Delta\varphi = 0$, 而单连通区域内调和函数的行为完全由它的边值决定.

注. 二维情况下, 我们把速度场视作 $\mathbb{R}^3$ 中第三分量为零的向量 $u = (u_1, u_2, 0)$, 此时 $\nabla\times u = (0, 0, \partial_2 u_1 - \partial_1 u_2)$. 我们定义其旋度为 $\omega := \nabla^\perp\cdot u = \partial_2 u_1 - \partial_1 u_2$.

2.1 Birkhoff-Rott 方程

为了叙述简便, 我们用复数 $z = x + iy$ 来表示 $\mathbb{R}^2$ 中的点 (x, y). $\bar{z} := x - iy$ 为 z 的复共轭, $f_x := \partial_x f$ 为 f 的偏导数. H^s 为 Sobolev 空间. 现在我们来寻求涡片的演化方程. 我们将流体的旋度视作支于曲线 (自由界面)$\Gamma(t)$ 上的一个 Radon 测度, 由位置函数 $\xi = \xi(s, t)$ 给出, 其中 s 为弧长参数, $\xi(0, t)$ 为某个液滴的路径. 这个观点由 Majda [19] 提出. 该曲线上, 旋度密度记作 $\gamma(s, t)$, 即对任意时间 t, 对任意 $\phi\in C_c^\infty(\mathbb{R}^2)$, 有

$$\iint \phi(x, y)\omega(x, y; t)dxdy = \int \phi(\xi(s, t))\gamma(s, t)ds.$$

根据流体的**毕奥-萨法尔定律** (Biot-Savart law), 给定边界上的旋度 $\xi(s, t)$, 我们可以将流体内部的速度场 u 还原出来, 它的复共轭等于下式

$$\forall z\notin\Gamma(t),\quad \bar{u}(z, t) = \frac{1}{2\pi i}\int \frac{\gamma(s', t)}{z - \xi(s', t)}ds'.$$

该式仅在自由界面 (涡片)$\Gamma(t)$ 处有不连续点出现. 涡片上的速度场则可以用涡片两侧的速度的平均值 (对流速度) 来定义

$$\bar{u}(\xi(s,t),t) = \frac{1}{2\pi i}P.V.\int \frac{\gamma(s',t)}{\xi(s,t)-\xi(s',t)}ds'.$$

此时, 我们可以得到涡片的演化方程 (具体推导以及它与欧拉方程组的等价性证明, 可以参见 [27])

$$\begin{cases}\xi_t(s,t)+a(s,t)\xi_s(s,t)=u(\xi(s,t),t)\\ \gamma_t(s,t)+\partial_s(a(s,t)\gamma(s,t))=0\end{cases} \tag{2.1}$$

其中 $a(s,t)$ 为一个实值函数, 满足 $a(0,t)=0$. 进一步假设 $\alpha(s,t)=\int_0^s \gamma(s',t)ds'$, 并作变量替换 $z(\alpha,t)=\xi(s(\alpha,t),t)$. 这里 $s(\alpha,t)$ 是 $\alpha(s,t)$ 的反函数: $\alpha(s(\alpha,t))=\alpha$. 那么(2.1)可以化作如下的 **Birkhoff-Rott 方程**

$$\bar{z}_t(\alpha,t)=\frac{1}{2\pi i}P.V.\int \frac{1}{z(\alpha,t)-z(\beta,t)}d\beta. \tag{2.2}$$

这个方程相当于以 "环量"(circulation)α 作为参数, 给出涡片的演化方程. $\gamma := 1/|z_\alpha|$ 称作**涡量** (vortex strength). 例如 $z=\alpha$ 这个定态解就是一个平坦涡片.

Birkhoff-Rott 方程的研究已经持续了几十年. 尽管之前已经提到, 在 Sobolev 空间 H^s 或者连续函数空间 C^γ 中, 我们可以构造不连续依赖初值的解来破坏掉局部适定性, 但我们仍可以在实解析函数空间中证明局部适定性, 甚至建立平坦涡片附近某类小解析初值的整体解存在性, 参见 [32, 4, 20]. 但是对某些解析初值, 涡片的曲率在有限时间内会演化出奇异性, 例如 Caflisch-Orellana [5] 构造了线性化方程在平坦涡片附近的反例 $z(\alpha,t)=S(\alpha,t)+r(\alpha,t)+\alpha$ (其中 $r \ll S$ 是一个可忽略的修正项)

$$S(\alpha,t) := \varepsilon(1-i)\left(\left(1-e^{-\frac{t}{2}-i\alpha}\right)^{1+\mu}-\left(1-e^{-\frac{t}{2}+i\alpha}\right)^{1+\mu}\right),\ 0<\varepsilon\ll 1,\ \mu>0.$$

实际上, 我们可以证明对任意的 $n>\mu$, $\partial^{1+n}S(\alpha,0)$ 在 $\alpha=0$ 附近会趋于无穷大.

2.2 弦-弧曲线与涡片解的存在性、实解析性

上述结果表明, 我们应当寻求一个尽可能大的函数空间, 使得 Birkhoff-Rott 方程在该函数空间里是适定的. 首先, 对固定的时间 t, 函数 $z(\alpha,t)\in L^2_{loc}(\mathbb{R})$, 而 Birkhoff-Rott 方程本身具有一个类似于柯西积分的形式, 从而可以用调和分析中的非卷积型奇异积分的 L^2 有界性定理来研究.

首先引入一个定义. 设有平面上以弧长参数 s 定义的 Jordan 曲线 $\Gamma:\xi=\xi(s)$, 我们称它为**弦-弧曲线**[1](chord-arc length), 是指存在常数 $M\ge 1$ 使得

$$|s_1-s_2|\le M|\xi(s_1)-\xi(s_2)|,\quad \forall s_1,s_2.$$

其中 M 可以取到的下确界称作**弦-弧常数**. 对这样的曲线, 我们可以证明 $\xi'(s)$ 几乎处处存在, 并且存在函数 $b\in BMO(\mathbb{R})$[2], 使得 $\xi'(s)=e^{ib(s)}$. 例如, 圆弧、以及类似于蜗牛壳的等角螺线 (极坐标方程为 $\rho=\pm e^\theta$) 就是这样的曲线. David [18] 证明了, 对这类曲线, 我们可以建立柯西积分算子的 L^2 有界性. 注意, 这与调和分析中非卷积型奇异积分的直接结论有所不同, 后者是对 Lipschitz 曲线证明的.

[1]这是字面翻译, 作者暂时没有查到这个单词的中文翻译, 《数学辞海》里也查不到.

[2]BMO 空间的定义为 $BMO(\mathbb{R}^d) := \left\{u\in L^1_{loc}\,\middle|\, \sup_{\text{正方体}Q}\frac{1}{|Q|}\int_Q |u(y)-u_Q|dy<\infty,\ u_Q := \frac{1}{|Q|}\int_Q u(y)dy\right\}$

引理 2.1 (David [18]). 对任一以弧长参数 s 定义的正则曲线 $\Gamma : \xi = \xi(s)$ (特别地, 弦-弧曲线也是正则曲线), 对应的柯西积分算子 C_Γ:

$$C_\Gamma f := P.V. \int \frac{f(s)}{\xi(s) - \xi(s')} d\xi(s')$$

是 $L^2(ds) \to L^2(ds)$ 的有界线性算子.

我们简要解释一下为什么需要这个引理. 实际上, 我们如果像(2.2)那样用毕奥-萨法尔定律, 通过边界上的旋度还原内部的速度场, 所得的积分很可能是发散的, 这是因为旋度在无穷远处未必有足够的衰减. 但是邬似珏 [40, 41] 首先注意到, 可以考虑对(2.2)取两个不同的位置作差, 即

$$\begin{aligned} \bar{z}_t(\alpha, t) - \bar{z}_t(\alpha', t) =& \frac{1}{2\pi i} P.V. \int_{|\beta| \le N} \left(\frac{1}{z(\alpha, t) - z(\beta, t)} - \frac{1}{z(\alpha', t) - \zeta(\beta, t)} \right) d\beta \\ &+ \frac{1}{2\pi i} P.V. \int_{|\beta| > N} \frac{z(\alpha', t) - z(\alpha, t)}{\big(z(\alpha, t) - z(\beta, t)\big)\big(z(\alpha', t) - \zeta(\beta, t)\big)} d\beta, \end{aligned} \tag{2.3}$$

其中 $N > |\alpha| + |\alpha'| + 1$. 在该设定下, 满足(2.2)的解必定满足(2.3), 而满足(2.3)的解 $z(\alpha, t)$ 若能使得(2.2)右边是收敛的, 则存在 $c(t)$ 使得 $z(\alpha, t) + c(t)$ 是(2.2)的解. 这个新的发现, 结合上述引理, 可以让某些使得(2.2)右边积分发散的特解, 其对应的(2.3)式是收敛的: 对几乎处处的 $(\alpha, t), (\alpha', t)$, (2.3)右边收敛, 且属于 $L^\infty\big([0, T]; L^2_{loc}(d\alpha) \times L^2_{loc}(d\alpha')\big)$.

定理 2.2 ([41, Theorem 1.2], 涡片解的实解析性). 设

$$z \in H^1\Big([0, T]; L^2_{loc}(\mathbb{R}) \bigcap L^2\big([0, T]; H^1_{loc}(\mathbb{R})\big)\Big)$$

且 z 是(2.2)在时间 $[0, T]$ 内满足如下条件的解:

(1) $\forall \alpha, \beta \in \mathbb{R}, 0 \le t \le T$, 有 $|z(\alpha, t) - z(\beta, t)| \simeq |\alpha - \beta|$.

(2) 对区间 (a, b), 存在与时间 t 无关的 $\delta_0 > 0$ 使得 $\sup_{0 \le t \le T} \|\ln z_\alpha(\cdot, t)\|_{BMO(a,b),\delta_0} \le C(M, m)$. 其中 m, M 为上一个不等式的上下界常数.

(3) $\ln z_\alpha \in L^2\big([0, T]; L^2_{loc}(\mathbb{R})\big)$. (进而涡量 $\gamma = 1/|z_\alpha|$ 在 $0, \infty$ 附近之外的点都是有界的)

则 $z_\alpha \in C\big((a, b) \times (0, T)\big)$, (且) 对每个固定的 $t_0 \in [0, T]$, $z_\alpha(\cdot, t_0)$ 是 (a, b) 上的实解析函数.

该定理表明, 在每个固定的时间点 t, 方程的解 $z(\cdot, t)$ 都是一条弦-弧曲线. 而上述定理的第一条假设相当于是 "涡片在 (a, b) 这一段不会过快地翻卷起来". 同时, 之前提到 [5] 构造的反例正是违背了上述定理的假设 (尤其是第一个假设) 才导致了涡片解在有限时间内就演化出奇异性: 在该奇异性发生之后的瞬间, 涡片解崩塌, 即涡片解不再是弦-弧曲线.

上述定理表明, 若涡片解存在, 且满足某些比较好的条件和某些 "小扰动" 条件, 则它是实解析的. 下面我们就进一步考虑涡片解的存在性问题: 给定初值, Birkhoff-Rott 方程是否一定有满足定理 2.2 条件的解？怎么样的函数才有资格成为 Birkhoff-Rott 方程的初值？实际上, 我们只需要像单个流体的欧拉方程那样给定流体的初速度 u_0 就够了. 其原因是: 流体的初速度 v_0 与旋度 ω_0 是相互决定的 (前者得到后者直接求导就行, 后者得到前者是通过毕奥-萨法尔定律), 而二维无旋流的涡片问题中的旋度在边界上是支于曲线的 Radon 测度, 它由涡片初始位置 $\xi(\cdot, 0)$ 和初始涡量 $\gamma(\cdot, 0)$ 共同决定, 而这两个量的信息都被包含在重新选取参数 α 后所得到的 $z(\alpha, 0)$ 里面.

定理 2.3 ([41, Theorem 1.4], 涡片解的存在性). 任给实值函数 $\omega_0 \in H^{1.5}(\mathbb{R})$, 存在时间 $T(\|\omega_0\|_{H^{1.5}}) > 0$ 使得方程(2.2)在 $[0, T]$ 内有解 $z(\alpha, t)$, 并满足

$$z \in H^1\Big([0, T]; L^2_{loc}(\mathbb{R}) \bigcap L^2\big([0, T]; H^1_{loc}(\mathbb{R})\big)\Big)$$

且 z 是(2.2)在时间 $[0, T]$ 内满足如下条件的解:

(1) $\ln z_\alpha \in C\big([0,T];H^{1.5}(\mathbb{R})\big)\bigcap C^1\big([0,T];H^{0.5}(\mathbb{R})\big)$, *(这个条件实际上蕴含了定理2.2的第三条假设.)*

(2) $Im\ \big((1+i)\ln z_\alpha(\alpha,0)\big)=\omega_0(\alpha)$,

(3) $\forall\alpha,\beta\in\mathbb{R},0\le t\le T$, *有* $|z(\alpha,t)-z(\beta,t)|\simeq|\alpha-\beta|$.

一般情况下, 若不对方程的初值加以如上所述的限制, 则初值演化出的解未必满足定理2.2中三个条件. 实际上, 定理2.3已经在这一点上做到了最佳结果. 简要地说, 如果初值不满足上述定理的某个条件, 则可能导致方程解的实解析性丢失, 进而方程的解将不再是弦-弧曲线.

定理 2.4 ([41, Theorem 1.5], 涡片解的初值). *设* $z\in H^1\Big([0,T];L^2_{loc}(\mathbb{R})\bigcap L^2\big([0,T];H^1_{loc}(\mathbb{R})\big)\Big)$ *且* z *是*(2.2)*在时间* $[0,T]$ *内满足定理2.2条件的解, 并假设* $\ln z_\alpha\in L^2\big([0,T];L^2_{loc}(\mathbb{R})\big)$, $\omega_0=Im\ \big((1+i)\ln z_\alpha(\alpha,0)\big)$ *是* (a,b) *上的实解析函数, 则必有* $z_\alpha\in C\big((a,b)\times(0,T)\big)$, *且* $Re\ \big((1+i)\ln z_\alpha(\alpha,0)\big)$ *也是* (a,b) *上的实解析函数.*

上述三个定理即为著名华人数学家邬似珏 [41] 的主要结果, 这篇文章的数学证明部分超过了 140 页, 因此证明过程将不在本文列出.

3 不可压缩流：其它物理量的稳定性作用

尽管在很多情况下, 涡片解如同 KHI 现象所述那样上是十分不稳定的, 但如果考虑某些其它物理量的影响, 我们仍然可以对涡片解的存在性、稳定性进行很深入的研究.

3.1 表面张力阻止 KHI

表面张力就是一个阻止 KHI 发生的因素, 这个现象由 Birkhoff [3] 提出. 在数值计算上这个现象首先被侯一钊等人 [23] 证实, 他们还注意到, 如果是无旋流, 则可能可以存在长时间的解. 数学上, 带表面张力的涡片问题 (在 Sobolev 空间中的) 局部适定性的严格证明首先由 Ambrose [1] 给出了二维无旋流的情况, 此后他与 Masmoudi 给出了三维无旋流的证明 [2].

实际上 [23, 1, 2] 的证明想法很相似, 他们注意到: 尽管涡片的法向速度必须要通过 Birkhoff-Rott 方程解出来, 但是其切向速度未必如此, 这是因为涡片的切向运动速度并不影响它的形状, 它的实质贡献只体现在曲线的参数化上. 据此, 我们把切向速度当作一个单独的变量 T, 并通过适当选取 T, 使得涡片曲线的弧长参数 s 与 α 无关 (也就是说, 不论涡片如何演化, 它始终是同一个弧长参数给出的曲线). 另一方面, 选取不同的切向速度变量会改变涡片曲线的演化方程, 尤其是在化简表面张力带来的曲率项时, 会出现包含 $(T-W\cdot\tau)$ 的项, 其中 W 是 Birkhoff-Rott 方程给出的速度场. 但如果我们引进拉格朗日坐标系的话, $(T-W\cdot\tau)$ 实际上会变成零. 余下的工作即是对演化方程作能量估计, 通过卷积光滑化构造逼近解并取极限, 而逼近方程则由 Picard 迭代法解出.

若旋度非零, 则我们不能再用 Birkhoff-Rott 方程来求解涡片问题, 这是因为旋度的存在使得流体的速度场无法写成向量势的梯度, 进而流体的运动不再只由边界运动决定. 这个时候, 就要对整个欧拉方程组的自由交界面问题(1.1)进行研究, 其局部适定性的证明由 Cheng-Coutand-Shkoller [11] 给出, 其构造解的方法与 Coutand-Shkoller [17] 关于带旋不可压缩流的局部适定性证明类似, 而能量估计的方法则来源于 Christodoulou-Lindblad [12] 先验估计. 这在上一期《蛙鸣》作者的文章里面已经有所介绍, 此处不再赘述. 而后, Shatah-曾崇纯 [30, 31] 也给出了一个新的证明.

数学上来看, 表面张力能提高自由界面的正则性. 若速度场在流体内部的正则性为 $H^r(\Omega(t))$, 那么表面张力可以使得自由界面的正则性至少达到 $H^{r+1}(\Gamma(t))$. 这也是数学上非

零表面张力情况能做出适定性的根本原因——无表面张力情况的边界正则性较低, 边界项失去控制. 具体而言, 我们不妨把两个流体间的自由界面 $\Gamma(t)$ 视作 $\mathbb{R}^3$ 中的嵌入子流形. 设嵌入映射为 $\eta:\Gamma(t)\to\mathbb{R}^3$, 则嵌入子流形的平均曲率由 $-\Delta_g(\eta|_{\Gamma(t)})=(\mathcal{H}\circ\eta)n$ 给出, 其中 g 为自由界面上由嵌入映射诱导出的度量 (具体表达式为 $g_{ij}=\partial_i\eta_\alpha\partial_j\eta^\alpha,\ i,j=1,2,\ \alpha=1,2,3.$) 这样, 在 H^r 的切向能量估计中, 自由界面的主要贡献项为

$$\sigma\int_\Gamma \sqrt{g}g^{ij}n^\alpha n^\beta\overline{\partial}^r\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}^r\partial_t\eta_\alpha \overset{\overline{\partial}_i}{\approx} -\frac{1}{2}\frac{d}{dt}\int_\Gamma\sqrt{g}g^{ij}\left|\overline{\partial}^r\overline{\partial}\eta\cdot n\right|_0^2+\cdots \tag{3.1}$$

而法向导数的估计则通过散度、旋度控制. 笔者与合作者在去年九月的文章 [22] 给出了计算自由界面表面张力的一个较为系统的方法, 并且该方法可以适用于弹性介质、磁流体等更复杂的流体模型. 特别地, 若将流体的自由界面视作一个函数图像 $\Gamma(t):=\{(x_1,x_2,x_3):x_3=\psi(t,x_1,x_2)\}$, 则表面张力的计算会简化很多, 因为此时自由界面的平均曲率可以直接显式地给出

$$\mathcal{H}=\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{1+|\overline{\nabla}\psi|^2}\right),$$

进而如上计算给出的能量项变为 $|\overline{\partial}^r\overline{\nabla}\psi|^2_{L^2(\Gamma)}$. 图像坐标的计算方法可以参考王焰金-辛周平 [38].

3.2 磁场的稳定效应

除了表面张力之外, 在一些更复杂的流体模型里面, 那些流体之外的物理量也可能阻止 KHI 的发生. 这里, 我们打算以磁流体方程组 (MHD) 为例, 简要地介绍磁场对涡片解的稳定性作用. 首先, 我们引入不可压缩的理想磁流体的涡片问题如下

$$\begin{cases}(\partial_t+u^\pm\cdot\nabla)u^\pm-(B^\pm\cdot\nabla)B^\pm=-\nabla P^\pm,\ P:=p+\frac{1}{2}|B|^2 & \text{in }\Omega^\pm(t),\\ \text{div }u^\pm=0 & \text{in }\Omega^\pm(t),\\ (\partial_t+u^\pm\cdot\nabla)B^\pm-(B^\pm\cdot\nabla)u^\pm=0 & \text{in }\Omega^\pm(t),\\ \text{div }B^\pm=0 & \text{in }\Omega^\pm(t).\end{cases} \tag{3.2}$$

其中 $B^\pm$ 是区域 $\Omega^\pm$ 内的磁场. 这个方程组描述的是两个互不渗透的理想磁流体的运动, 其自由交界面 $\Gamma(t)$ 被称作**电流涡片** (current-vortex sheet), 在自由界面上我们有跳跃边界条件件

$$u^+\cdot n=u^-\cdot n=\Gamma(t)\text{的运动速度},\quad \text{(动力学边界条件)} \tag{3.3}$$

$$[P]:=P^+-P^-=\sigma\mathcal{H},\quad \text{(压力平衡条件)} \tag{3.4}$$

$$B^+\cdot n=B^-\cdot n=0.\quad \text{(理想导体条件)} \tag{3.5}$$

其中 n 为 $\Gamma(t)$ 关于 $\Omega^+(t)$ 一侧的单位外法向量.

注 . *方程 $div\ B=0$ 以及边界条件(3.5)均不是独立的方程 (否则方程个数比未知数多, 原方程组超定), 它们都只是对初值的限制: 若初值满足这两个条件, 则解在任意时刻都满足这两个条件.*

注 . *边界条件(3.3)-(3.5)表明, 流体的速度场、总压力、磁场均在自由界面的法方向连续, 但在切方向可能存在间断. 特别地, 因为速度场存在切向间断 $[u_\tau]\neq\mathbf{0}$, 这个自由界面被称作涡片; 因为磁场也存在切向间断 $[B_\tau]\neq\mathbf{0}$, 自由界面上会产生表面电流 $j^*:=[B]\times n$, 从而交界面被称作"电流涡片".*

电流-涡片模型又称作等离子体-等离子体自由界面问题, 其在物理中有直接应用 (当然主要都是用可压缩流的情况): 一方面, 该模型被用于等离子体的磁约束模型, 这会出现在聚

变过程中 (特别地, 若被包围的等离子体是液态金属, 则表面张力的作用是不可以忽略的); 另一方面, 在天体物理中, 电流涡片实际上是 “磁层顶” 的数学刻画, 例如太阳系的 “日球层顶”(heliopause), 其被视作太阳系的理论边界. 当太阳风 (大量等离子体) 从太阳被发射出来时是一股超音速流, 当其抵达太阳系边界附近的终端激波后, 超音速流降为亚音速流 (因此出现了跨音速磁流体激波), 并被压缩, 直至其 “抵达” 日球层顶, 后者将太阳风的等离子体与太阳系之外压缩在弓状激波附近的等离子体 “隔开”(恰好对应了边界条件(3.5)).

当表面张力系数 $\sigma = 0$ 时, 我们需要在自由界面 $\Gamma(t)$ 上加一个额外条件 (实际上也是对初值的限制, 若初值成立, 则该条件至少能保持一小段时间), 简单地说就是**两个磁场 $B^{\pm}$ 在自由界面 $\Gamma(t)$ 上处处不平行**. 这个条件被称作 Syrovatskiĭ 条件, 具体形式如下

$$|B^{\pm} \times [u]|^2 + |B^{\pm} \times [u]|^2 < 2\,|B^{+} \times B^{-}|^2 . \tag{3.6}$$

这个条件早在 1953 年被 Syrovatskiĭ 提出: 当不考虑表面张力时, 若没有条件(3.6), 则如上问题对应的线性化方程的解会瞬间爆破. 类似的结论可以在 Landau-Lifshiz-Pitaevskiĭ 的《连续介质电动力学》[24] 中查到. 数学上, Yuri Trakhinin [36] 首先严格证明了这一条件的必要性, Morando-Trakhinin-Trebeschi [29] 用 “双曲对称化” 的方法得到了线性化方程解的存在性. 而对非线性问题 (3.2)-(3.6) 的求解则是更困难的: 在得到线性化方程的解的基础上, 我们还需要一个合适的迭代方法以得到非线性方程的解, 还要对得到的解建立非线性的估计. 首个非线性估计由 Columbel-Secchi-Trebeschi [13] 得到, 但是非线性问题的求解仍是困难的, 因为迭代过程中有一个关键的消去结构不再成立, 进而会损失一阶切向导数. 后来, 孙永忠-王伟-章志飞 [34] 通过推广 Shatah-曾崇纯 [30] 的方法, 并借用邬似珏 [39] 的一个想法, 最终证明了非线性问题的适定性, 即证明了电流涡片在 Syrovatskiĭ 条件下的非线性稳定性.

数学上来看, Syrovatskiĭ 条件可以将边界正则性提高到 $H^{r+0.5}(\Gamma(t))$, 即只比表面张力的情况低 0.5 阶正则性. 而在没有表面张力时, 平均曲率这个带有二阶导数的项不再出现, 因此 Syrovatskiĭ 条件带来的增益已经足够让我们控制边界项的贡献. 这多出来的 0.5 阶正则性可以通过**对边界运动方程 (即动力学边界条件(3.3)) 求一阶时间导数**证得. 为了方便, 我们不妨假设自由界面是函数图像 $\Gamma(t) := \{(x_1, x_2, x_3) : x_3 = \psi(t, x_1, x_2)\}$, 此时的(3.3)变为

$$\partial_t \psi = u \cdot N, \quad N = (-\partial_1 \psi, -\partial_2 \psi, 1)^{\top}.$$

对该方程求时间导数, 右边就会出现 $\partial_t u \cdot N$ 项, 这个时候把(3.2)中的第一个方程代进来替换掉这项, 经过繁琐的计算, 可以最终得到

$$D_t^2 \psi - \underbrace{\left(\frac{1}{2} B_i^+ B_j^+ - B_i^- B_j^- - [u]_i [u]_j\right)}_{\Lambda_{ij}} \partial_i \partial_j \psi = \cdots , \quad i, j = 1, 2, \tag{3.7}$$

其中 $D_t = \partial_t + w_1 \partial_1 + w_2 \partial_2$, $w = \frac{1}{2}(u^+ + u^-)$ 为对流速度. 而 **Syrovatskiĭ 条件蕴含了 (Λ_{ij}) 的正定性**. 这样, 边界运动方程就有类似于波方程的形式. 其能量估计可以通过两边求 $\overline{\partial}^{r-1/2}$ 并分部积分一次得到:

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Gamma(t)} \left| \overline{\partial}^{r-1/2} D_t \psi \right|^2 + \left| \overline{\partial}^{r-1/2} \overline{\partial} \psi \right|_0^2 \right) \lesssim \overline{\partial}^{r-1/2} \Big((3.7)\text{的右边} \Big),$$

该式的右边可以用 Sobolev 迹定理控制, 化为各个量内部的 H^r 范数. 这就完成了最关键的一环.

另一方面, 若 $\sigma > 0$, 人们期待的结果是去掉 Syrovatskiĭ 条件也能证得适定性. 数学上来看, 表面张力能带来的边界正则性已经比 (3.6) 高出 0.5 阶. 目前, 这方面还没有一个正确的严格数学证明, 笔者也期待与合作者在这方面有所建树.

4 可压缩流：超音速条件与附加物理量的共同影响

若考虑流体的可压缩性, 即密度不为常数时, 欧拉方程变成如下形式

$$\begin{cases} \rho^{\pm}(\partial_t + u^{\pm}\cdot\nabla)u^{\pm} = -\nabla p^{\pm} & \text{in } \Omega^{\pm}(t), \\ (\partial_t + u^{\pm}\cdot\nabla)u^{\pm} + \rho^{\pm}\mathrm{div}\ u^{\pm} = 0 & \text{in } \Omega^{\pm}(t), \\ \rho^{\pm}(\partial_t + u^{\pm}\cdot\nabla)S^{\pm} = 0 & \text{in } \Omega^{\pm}(t), \end{cases} \tag{4.1}$$

其中 $S^{\pm}$ 为熵 (entropy). 边界条件仍为

$$u^{+}\cdot n = u^{-}\cdot n = \Gamma(t)\text{的运动速度}. \quad \text{(动力学边界条件)} \tag{4.2}$$

$$[p] := p^{+} - p^{-} = \sigma\mathcal{H}. \quad \text{(压力平衡条件)} \tag{4.3}$$

对可压缩流, 我们还需要明确**状态方程**, 即流体压力 $p = p(\rho, S)$ 是密度和熵的函数, 且关于密度 ρ 单调递增. 为了方便, 我们只考虑等熵、液体的情况, 即假设 (1)S 为常数, (2) $\rho^{\pm}$ 有严格正的下界. 其中条件 (1) 去掉也是对的 (只是证明麻烦一点), 条件 (2) 是必需的, 因为它保证了方程组作为一阶双曲组的**严格双曲性**, 在边界没有退化.

4.1 超音速条件的必要性

相比不可压缩流而言, 可压缩流里面多出了 “声波” 的贡献 (对第一个方程求散度并代入第二个方程就得到一个 p 的波方程). 对一个速度场为 v, 密度为 ρ, 压力为 p 的可压缩流体, 我们定义该流体中的**声速**为 $c := \sqrt{\partial p/\partial\rho}$, 定义流体的**马赫数** (Mach number) 为 $M := |v|/c$. 根据第 2 节的讨论, 我们知道当 $\sigma = 0$ 时, 不可压缩流的涡片问题在 Sobolev 空间里是连局部适定性都没有的. 但是当流体可压缩时, 即便 $\sigma = 0$, 我们仍可以对二维的情况, 在马赫数 $M > \sqrt{2}$ 的定态解 $\mathring{U}$ 附近建立 Sobolev 空间中的局部适定性 (即 $U - \mathring{U} \in H^r$, 而非 $U \in H^r$. 这里 $\mathring{U}$ 是给定的定态解, 马赫数大于 $\sqrt{2}$, 一般来说我们在 $\Omega^{\pm}$ 中取 $\mathring{U}$ 的速度分量为 $(\pm\mathring{v}, 0)$, 并且该速度 $\mathring{v}$ 满足 $|\mathring{v}| > \sqrt{2}c$), 但在三维无法做出类似结果 [21].

二维流体的 “超音速条件”$M > \sqrt{2}$ 最早由 Miles [28] 提出. 数学上, Coulombel-Secchi [14] 在 2004 年证明了该条件对线性化方程解的存在性是必要的. 其证明方法是将整个方程视作带有**特征边界条件**的一阶严格双曲方程组, 在研究其线性化问题时, “超音速条件” 保证了边界运动方程的二阶导数项前面的系数有固定符号和下界

$$\left(\partial_t + \frac{v_1^{+} + v_1^{-}}{2}\right)^2 \psi + \frac{[v_1]^2}{4}\overline{\partial}^2\psi + \frac{c^2}{2}\nabla(p^{+} + p^{-})\cdot N = \cdots,$$

该方程可以通过一些傅立叶分析的方法得到能量估计. 而线性化的问题解决后, 为了进一步求解非线性问题, 我们需要进行合适的迭代. 2008 年, Coulombel-Secchi [15] 利用 Nash-Moser 迭代的方法证明了非线性问题的适定性.

但在 $\sigma > 0$ 时, 表面张力已经可以给边界带来 H^{r+1} 的正则性, 因此超音速条件是可以去掉的. 涡片的非线性稳定性在二维、三维的情况都可以证得. Stevens [33] 推广了 [16] 的抛物正则化方法, 完成了这一证明, 全文长达 128 页.

4.2 可压缩磁流体

对可压缩磁流体而言, 即便超音速条件 $M > \sqrt{2}$ 不成立, 但只要 Syrovatskiĭ 条件(3.6)成立, 那么电流涡片的非线性稳定性一定是成立的. 这个结果由 Yuri Trakhinin [37], 陈贵强-王亚光 [6] 分别独立做出.

尽管和不可压缩 MHD 的结论相同, 但是其证明方法、使用的函数空间都和不可压缩的

情况有非常大的差别. 我们来看可压缩 MHD 方程组的形式

$$\begin{cases}\rho^\pm(\partial_t+u^\pm\cdot\nabla)u^\pm-(B^\pm\cdot\nabla)B^\pm=-\nabla P^\pm,\ P:=p+\frac{1}{2}|B|^2 & \text{in }\Omega^\pm(t),\\ (\partial_t+u^\pm\cdot\nabla)u^\pm+\rho^\pm\mathrm{div}\ u^\pm=0 & \text{in }\Omega^\pm(t),\\ (\partial_t+u^\pm\cdot\nabla)B^\pm-(B^\pm\cdot\nabla)u^\pm=\underline{-B^\pm\mathrm{div}\ u^\pm} & \text{in }\Omega^\pm(t),\\ \mathrm{div}\ B^\pm=0 & \text{in }\Omega^\pm(t).\end{cases}\tag{4.4}$$

在自由界面上我们有跳跃边界条件

$$u^+\cdot n=u^-\cdot n=\Gamma(t)\text{的运动速度},\quad (\text{动力学边界条件})\tag{4.5}$$

$$[P]:=P^+-P^-=\sigma\mathcal{H},\quad (\text{压力平衡条件})\tag{4.6}$$

$$B^+\cdot n=B^-\cdot n=0.\quad (\text{理想导体条件})\tag{4.7}$$

之前我们提到, 不可压缩流的法向导数可以由散度、旋度控制. 但对可压缩 MHD 而言, 其旋度没有办法得到控制, 原因在于(4.4)中的磁场演化方程里出现了下划线项 $B\mathrm{div}\ u$. 这一项无论是对欧拉方程 ($B=\mathbf{0}$) 还是不可压缩 MHD(div $u=0$) 而言都不出现, 但恰恰是这一项导致了旋度估计的失败. 实际上, 在导出(4.4)的 L^2 能量守恒式的过程中, 这一项的贡献与压力项 $-\nabla P$ 的一部分抵消了. 但只要对原方程求旋度, 那么 $\nabla\times\nabla P=\mathbf{0}$, 而 $\nabla\times(B\nabla\cdot u)$ 却是一个包含了二阶导数的项, 这引起了法向导数的损失. 笔者曾经试过给磁场加耗散项再取极限的方法. 尽管耗散项恰好弥补了导数损失, 笔者据此证明局部适定性 [42], 但是无法做关于耗散系数一致的估计, 因此无法取得零耗散极限.

从另一个角度看, 可压缩 MHD 的涡片问题是一阶双曲组带特征边界条件的自由界面问题. 对于带特征边界条件的严格双曲组, 陈恕行先生在 1982 年 [10] 引入了所谓的各向异性 Sobolev 空间 H_*^m 来解决法向导数损失的问题. 不妨假设区域是半空间 $\mathbb{R}^2\times(0,\infty)$. 我们将切向导数 ∂_1,∂_2 以及带权法向导数 $x_3\partial_3$ 视作一阶导数, **将法向导数 ∂_3 视作二阶导数**. 实际上, 对这类双曲组, 我们**每约化掉边界上的一阶法向导数时, 都需要两阶切向导数作为补偿**, 而引进的 "各向异性" 恰好将一阶法向导数和二阶切向导数设置成同阶导数.

利用这类函数空间, 我们可以用 Lax-Phillips 对偶的方法 (参见 [25]) 证明可压缩 MHD 电流涡片的线性化问题的可解性. 但是对自由边界问题而言, 迭代过程中会产生导数损失, 因此目前的结果 [6, 37] 都是依赖于 Nash-Moser 迭代所得. 而用 Nash-Moser 迭代构造的解, 其光滑性比初值相差很多. 例如 [37] 的结果要求初值在 H_*^{2m+19} ($m\geq 12$), 得到的局部解只能落在 H_*^m 中, 可见导数损失的阶数至少有 31 阶, 并且 m 越大, 解的光滑性相对初值而言损失越多.

4.3 未解决的问题

对于可压缩流体的涡片问题, $\sigma>0$ 的情况除了 [33] 之外也没有其它任何结果. 因此, 声波和表面张力的共同作用对可压缩流的涡片稳定性的影响究竟如何, 至今仍是未解之谜. 而 [33] 所用的抛物正则化方法引入过多粘性项, 整个证明无比复杂, 而且这个方法无法推广到欧拉方程之外的流体模型, 也难以做到无界区域上 (解线性化方程时用到了 $-\Delta$ 的谱是离散的). 因此, 能否找到一个简洁的通法, 来得到带表面张力的可压缩涡片的非线性稳定性, 是一个值得深思的问题.

另一方面, 除了 Stevens [33] 之外, 其它所有非线性稳定性的结果都依赖于 Nash-Moser 迭代, 从而所得到的解相比初值而言都有严重的正则性损失. 笔者在博士期间的研究中, 已经发现了新方法能避开单个流体的可压缩 MHD 自由边界问题的导数损失, 参见 [26], 该结果是往最优估计这个方向前进的首个突破. 其证明关键是引入 "**带修正项的 Alinhac 好变量**" 来避免内部计算和各向异性共同带来的法向导数损失, 造出约化结构以消去因自由边界

带来的额外误差项, 而在边界项的处理上巧妙地利用各向异性空间的特点, 最后完成不损失正则性的能量估计. 笔者目前正在设法推广 [26] 到可压缩 MHD 的电流涡片问题, 以改进 [6, 37] 的结果, 做到最优估计; 并设法结合 [22] 做到非零表面张力的情况.

进一步, 我们可以设法证明 "不可压缩极限"——当流体的可压缩性趋于零时 (声速 $c \to \infty$), 若可压缩涡片的初值收敛到不可压缩涡片的初值, 则可压缩涡片解也收敛到对应的不可压缩涡片解, 这样就相当于证实了磁场对不可压缩流 KHI 的稳定效应可以 "移植到" 可压缩流的情况. 这相比单个流体的不可压缩极限而言是难度陡增的, 因为涡片的边界条件是**跳跃条件**, 而不是对 P^+, P^- 单独设置的条件. 在这方面, 目前没有任何数学上的结果, 这也是笔者想要探索的一个问题.

此外, 对其它的流体模型中的涡片问题, 在不考虑表面张力的情况下, 边界上的稳定性条件也各不相同. 例如弹性流体 (elastodynamics, 用于描述弹性波在介质中的传播, 例如海洋中的地震波) 的稳定性条件只能对应马赫数小于 $\sqrt{2}$ 的一部分区域, 其对定态解的声速仍有一定要求, 具体可以参见以陈明、王德华为首的研究团队对二维弹性介质涡片解的研究 [7, 8, 9]. 三维的情况, 以及表面张力非零的情况, 目前也都是未知的.

而可压缩涡片问题的长时间存在性, 即便是无旋情况, 也毫无进展. 实际上, 即便是可压缩无旋水波的长时间适定性, 目前也是未解决的. 如何化解自由边界对衰减估计的阻碍, 如何控制自由边界运动和内部声波之间的相互作用, 人们或许需要一些全新的想法来攻克这些难题. 毕竟, 这些问题不像不可压缩无旋水波那样可以写成边界上的色散方程, 从而用调和分析的工具就可以作出很精准的衰减估计以及长时间解.

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波方程局部适定性简介

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摘 要

我们简要介绍波方程的能量估计以及用能量估计来获得方程的局部适定性的方法.

1 介绍

为了让文章能面向更多的读者, 笔者尽可能用数学分析的语言框架来完成写作. 在引入更高级知识的时候, 会给出数学分析框架下的直观. 本篇文章有大二微分方程课程知识的同学即可阅读. 在文章的这一部分, 笔者从大二的课程知识出发, 引出我们在这篇文章里考虑的问题.

在大二的课程中, 我们考虑如下经典波方程

$$-\partial_t^2 u+\sum_{i=1}^{n}\partial_i^2 u=0,$$
$$u(0,x)=u_0,\ \partial_t u(0,x)=u_1.$$

在微分方程课程中, 我们学习了波方程的能量方法, 即在方程两边同时乘上 $\partial_t u$ 并关于空间积分. 在课程的学习中, 我们也见过能量估计的威力, 利用能量估计, 我们可以获得上述方程解的唯一性以及有限传播速度等性质.

1.1 更一般的波方程

现在我们考虑更一般化的波方程. 我们观察我们所熟知的波方程形式 $-\partial_t^2+\sum_{i=1}^n\partial_i^2$, 借助矩阵, 这个方程可以写成 $-\partial_t^2+\sum_{i,j=1}^n g_{ij}\partial_{ij}$, 其中 g_{ij} 是单位阵的元素. 现在我们不再假设这里的 $g=(g_{ij})$ 是单位矩阵, 我们允许 g 是任何一个正定对称的矩阵. 摆脱了单位阵的束缚, 自然也不必要求 g 是一个恒定矩阵, g 可以是一个随着 (t,x) 变化而变化的矩阵函数 $g(t,x)=(g_{ij}(t,x))$. 此外, 我们还需要假设 g 的正定性关于 (t,x) 满足一定的一致性, 即所谓的一致椭圆条件

$$\lambda|\xi|^2\le\sum_{i,j=1}^{n}g_{ij}(t,x)\xi_i\xi_j\le\Lambda|\xi|^2.$$

注. 这里对 g 从单位阵到正定对称矩阵的推广事实上来自于几何. 在黎曼几何中, 我们对某一个点赋予一个度量结构. 在选定局部坐标下, 这个度量结构由一个正定对称的矩阵来描述, 不同的点对应了不同的矩阵. 更一般地, 我们可以丢掉度量矩阵的正定性条件, 假设度量矩阵在正交相似下的标准形是 $(-,+,\cdots,+)$ 的, 即产生了 *Lorentz* 几何. 所以我们可以假设我们的方程是 $\sum_{\alpha,\beta=0}^n g_{\alpha\beta}(t,x)\partial_{\alpha\beta}u=0$, 其中 $\partial_0=\partial_t$ 且 $(g_{\alpha\beta})$ 是一个对称阵且 $(g_{\alpha\beta})$ 在正交相似下的标准形对角线是 $(-,+,\cdots,+)$ 的.

注. 波方程来源于物理. 在电磁学里我们就知道光可以分解为垂直的磁场 B 和电场 E, 解耦 *Maxwell* 方程我们知道 B 和 E 满足最经典的波方程. 更深刻地, 广义相对论中大名鼎鼎的 *Einstein* 方程就是一个波方程组. 对于波方程的研究, 可以促进对于物理现象的认识和理解.

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稿件信息: 4 月 9 日来稿, 5 月 23 日第一次修改, 5 月 25 日接受. 稿件编辑: 王子达.

1.2 局部适定性

在常微分方程的学习中, 除了求解各种各样的常微分方程, 我们还关注了常微分方程的抽象理论. 比如我们在什么条件下能保证常微分方程的初值问题的解是存在唯一的, 解的延拓性和对初值的连续依赖性. 在 PDE 中, 我们同样关心这些问题. 我们考虑初值问题

$$\begin{aligned} \Box_g u = -\partial_t^2 u + \sum_{i,j=1}^{n} g_{ij}(t,x)\partial_{ij} u = F, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x). \end{aligned} \tag{1.1}$$

我们称方程 (1.1) 是（局部）适定的, 若方程的解（局部）存在唯一且对初值连续依赖. 在这篇文章中, 我们主要讨论的就是线性和非线性波方程的局部适定性.

1.3 Sobolev 空间

Sobolev 空间是进一步学习 PDE 的必要工具. 在很多时候, 对函数的可导性是一个过分严苛的要求. 对可导性的过分执着可能会导致方程的解不存在. 所以我们要引入 Sobolev 空间, 弱化所谓“求导”的意义, 在更弱的意义下去讨论导数. 在引入 Sobolev 空间的具体定义之前, 我们先来看经典波方程的能量估计

$$\begin{aligned} \int_{x\in\mathbb{R}^n} (-\partial_t^2 u + \sum_{i=1}^{n} \partial_{i=i}^{n}\partial_i^2 u)\partial_t u dx &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} (\partial_t u)^2 dx + \int_{\mathbb{R}^n} \partial_i(\partial_i u \partial_t u) - \partial_i u \partial_{it} u dx \\ &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} (\partial_t u)^2 + \sum_{i=1}^{n} (\partial_i u)^2 dx. \end{aligned}$$

我们假设 u 是足够光滑的且 u 和 u 的空间导数在无穷远处趋于 0. 上面的最后一个等号来自于散度定理. 可以看到, 经典意义下的导数存在性并不关键, 我们更关心的是等式

$$\int_{\mathbb{R}^n} \partial_i u \phi dx = -\int_{\mathbb{R}^n} \partial_i u \phi dx, \quad \phi \in C_c^\infty(\mathbb{R}^n)$$

是否成立. 所以我们引入下面定义

定义 1.1. *我们称 u 的 α 阶弱导数存在, 即在弱意义下, 我们有 $\partial^\alpha u = v$, 是指对于任意的 $\phi \in C_c^\infty$, 我们有*

$$\int_{\mathbb{R}^n} u \partial^\alpha \phi dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \phi dx.$$

这里我们对于求导的记号 ∂^α 和 Evans 附录中一致, 不熟悉的读者可以去查阅 [3] 附录. (注意, 为了书写的方便, 我们又记 $\partial_t = \partial_0$).

定义 1.2. *我们称 u 属于 Sobolev 空间 $H^k(\mathbb{R}^n)$, 若对于任意的 $|\alpha| \le k$, u 的 α 阶弱导数存在且属于 $L^2(\mathbb{R}^n)$. 我们定义 u 的 k 阶 Sobolev 范数为*

$$\|u\|_{H^k(\mathbb{R}^n)} := \sum_{|\alpha|\le k} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^n)}.$$

Sobolev 空间有很多好的性质, 限于篇幅不在这里介绍. 事实上, 尽管弱导数的概念十分抽象且对函数的要求很低 (在谈论弱导数时我们甚至不需要连续条件), 但是当 $k > \frac{n}{2}$ 时, 事实上我们可以证明这时 H^k 中的函数是连续的. 详细的讨论可以参考 Evans 书本第五章. 为了方便我们的估计, 我们介绍一个 Sobolev 嵌入定理.

定理 1.1. *对任意的 $k > \frac{n}{2}$, 存在 $C = C(n,k)$ 使得*

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \le C\|\phi\|_{H^k(\mathbb{R}^n)}.$$

弱导数的概念启发我们从积分的角度理解方程, 同样的, 我们可以定义出方程的弱解, 从此我们就可以完完全全用弱意义去理解方程了

定义 1.3. 我们称 u 是方程 (1.1) 在 $[0,T]\times\mathbb{R}^n$ 的弱解, 若 u 满足

$$\int_0^T\int_{\mathbb{R}^n}F\varphi dxdt=\int_0^T\int_{\mathbb{R}^n}u\big(-\partial_t^2\varphi+\sum_{i,j}\partial_{ij}(g_{ij}\varphi)\big)dxdt$$
$$+\int_{\mathbb{R}^n}\varphi(0,x)u_1(x)dx-\int_{\mathbb{R}^n}\varphi_t(0,x)u_0(x)dx,\quad \forall\varphi\in C_c^\infty\big((-\infty,T)\times\mathbb{R}^n\big).$$

注. *弱解的定义只作了解, 后续不会用到.*

2 一般波方程的能量估计

证明方程 (1.1) 局部适定性的关键, 就是导出合适的能量估计. 首先我们回顾 Gronwall 不等式.

引理 2.1. *令 E,A,b 是定义在 $[0,T]$ 上的非负连续函数且 A 单调递增. 若我们有*

$$E(t)\le A(t)+\int_0^t b(\tau)E(\tau)d\tau,\quad 0\le t\le T,$$

则我们有估计

$$E(t)\le A(t)e^{\int_0^t b(\tau)d\tau}.$$

注. *Gronwall 不等式在如下结构的微分不等式*

$$\frac{d}{dt}E(t)\le b(t)E(t)$$

中会发挥很大作用. 在上式两边同时做积分, 我们就可以得到 Gronwall 不等式的条件式, 从而可以应用 Gronwall 不等式估计 $E(t)$.

方程 (1.1) 的能量估计思路与 $-\partial_t^2+\sum_{i=1}^n\partial_i^2$ 型的波方程证明思路相同, 我们在方程两边同时乘上 $\partial_t u$ 再对空间做积分, 利用求导的乘法法则和散度定理进行化简. 具体说来, 我们有如下的定理

定理 2.2. *对任意的 $u\in C^2\big([0,T]\times\mathbb{R}^n\big)$, 有*

$$\|\partial u\|_{L^2(\mathbb{R}^n)}(t)\le C\Big(\|\partial u\|_{L^2(\mathbb{R}^n)}(0)+\int_0^t\|\Box_g u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\Big)$$
$$\times e^{C\int_0^t\sum_{i,j=1}^n\|\partial g_{ij}(\tau,\cdot)\|_{L^\infty}d\tau},$$

其中 $|\partial u|^2=(\partial_t u)^2+\sum_{i=1}^n(\partial_i u)^2$, C 是只和 λ,Λ 有关的常数.

证明. 对 $\Box_g$ 乘上 $\partial_t u$ 并对空间做积分有

$$\int_{\mathbb{R}^n}\Box_g u\partial_t udx=\int_{\mathbb{R}^n}-\partial_t^2u\partial_t u+\sum_{i,j=1}^n g_{ij}\partial_{ij}u\partial_t udx$$
$$=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}(\partial_t u)^2dx+\sum_{i,j=1}^n\int_{\mathbb{R}^n}\partial_i(g_{ij}\partial_j u\partial_t u)-g_{ij}\partial_j u\partial_{it}u-\partial_i(g_{ij})\partial_j u\partial_t udx.$$

在整篇文章中, 我们总是假设 u 在空间无穷远处 (即 $|x|\to\infty$ 时) 迅速衰减趋于 0, 从而当我们看到和 u 有关的全空间散度型积分的时候, 我们可以认为这一项是 0, 因此我们有

$$\int_{\mathbb{R}^n}\Box_g u\partial_t udx=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}(\partial_t u)^2+\sum_{i,j=1}^n g_{ij}\partial_i u\partial_j udx+\sum_{i,j=1}^n\int_{\mathbb{R}^n}\frac{1}{2}\partial_t(g_{ij})\partial_i u\partial_j u-\partial_i(g_{ij})\partial_j u\partial_t udx.$$

应用 Holder 不等式, 我们有

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}(\partial_t u)^2+\sum_{i,j=1}^n g_{ij}\partial_i u\partial_j u dx \le \Big(\int_{\mathbb{R}^n}(\Box_g u)^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^n}(\partial_t u)^2dx\Big)^{\frac{1}{2}} + C\sum_{i,j=1}^n \|\partial g_{ij}\|_{L^\infty(\mathbb{R}^n)}(t)\int_{\mathbb{R}^n}(\partial_t u)^2+\sum_{i=1}^n(\partial_i u)^2dx.$$

由于 g 满足一致椭圆条件, 从而存在只和 λ,Λ 有关的常数 C_1, 使得

$$\frac{1}{C_1}\Big((\partial_t u)^2+\sum_{i=1}^n(\partial_i u)^2\Big)\le(\partial_t u)^2+\sum_{i,j=1}^n g_{ij}\partial_i u\partial_j u\le C_1\Big((\partial_t u)^2+\sum_{i=1}^n(\partial_i u)^2\Big).$$

从而有

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^n}(\partial_t u)^2+\sum_{i,j=1}^n g_{ij}\partial_i u\partial_j u dx\Big)^{\frac{1}{2}}\le C\Bigg(\Big(\int_{\mathbb{R}^n}(\Box_g u)^2dx\Big)^{\frac{1}{2}}+\sum_{i,j=1}^n\|\partial g_{ij}\|_{L^\infty(\mathbb{R}^n)}\Big(\int_{\mathbb{R}^n}(\partial_t u)^2 + \sum_{i,j=1}^n g_{ij}\partial_i u\partial_j u dx\Big)^{\frac{1}{2}}\Bigg).$$

上式两边同时在 $[0,T]$ 做积分就会产生 Gronwall 不等式所需要的结构, 从而可以应用 Gronwall 不等式完成估计. □

注. *从上面的证明过程我们可以看出, 我们考虑的更一般的波方程的能量估计和课上所学的经典波方程的能量估计的不同之处来源于这里的 g_{ij} 不再是一个常数, 因而我们需要处理对 g_{ij} 求导所产生的项.*

事实上, 上面获得的能量估计对于我们所考虑的局部适定性问题是不够的. 我们还需要所谓的高阶能量估计, 感兴趣的读者可以试试看在经典波方程中先对方程两边同时求 ∂_i, 然后再乘上 $\partial_t\partial_i u$, 重复能量估计的方法, 看看能得到什么结论. 对于我们考虑的更一般的波方程, 我们有下面的高阶能量估计.

定理 2.3. *设 $0<T<\infty, u\in C_c^\infty\big([0,T]\times\mathbb{R}^n\big), k\in\mathbb{Z}_+$. 假设 g_{ij} 及其各阶导数有界, 我们有*

$$\sum_{|\alpha|\le 1}\|\partial^\alpha u(t,\cdot)\|_{H^k}\le C\Big(\sum_{|\alpha|\le 1}\|\partial^\alpha u(0,\cdot)\|_{H^k}+\int_0^t\|\Box_g u(\tau,\cdot)\|_{H^k}d\tau\Big),\quad 0<t<T,$$

其中 C 是一个依赖于 T,k 以及 g_{ij} 的各阶导数的常数.

证明. 首先我们考虑 $k=0$ 的情况. 在上一个引理中我们已经证明了

$$\|\partial u\|_{L^2}(t)\le C\Big(\|\partial u\|_{L^2}(0)+\int_0^t\|\Box_g u\|_{L^2}(\tau)d\tau\Big).$$

我们还需要对 $\|u\|_{L^2}$ 做估计. 由微积分基本定理我们有

$$\begin{aligned}\|u(t,x)\|_{L^2}&=\|u(0,x)+\int_0^t\partial_t u(\tau,x)d\tau\|_{L^2}\\&\le\|u(0,\cdot)\|_{L^2}+\int_0^t\|\partial_t u(\tau,\cdot)\|_{L^2}d\tau\\&\le\|u(0,\cdot)\|_{L^2}+(1+t)\Big(\|\partial u\|_{L^2}(0)+\int_0^t\|\Box_g u\|_{L^2}(\tau)d\tau\Big),\end{aligned}$$

从而我们完成了 $k=0$ 情况的证明. 对于 $k\ge1$ 的情形, 我们先对方程两边同时对空间求 α 阶导数 $(|\alpha|\le k)$, 我们有

$$\partial_x^\alpha\Box_g u=-\partial_t^2\partial_x^\alpha u+\sum_{i,j=1}^n g_{ij}\partial_{ij}\partial_x^\alpha u+\sum_{i,j=1}^n[\partial_x^\alpha,g_{ij}]\partial_{ij}u,$$

$$\partial_x^\alpha \Box_g u - \sum_{i,j=1}^n [\partial_x^\alpha, g_{ij}]\partial_{ij}u = -\partial_t^2\partial_x^\alpha u + \sum_{i,j=1}^n g_{ij}\partial_{ij}\partial_x^\alpha u.$$

我们对上面最后一个式子应用 $k=0$ 时的能量估计, 我们有

$$\sum_{|\alpha|\le k,|\beta|\le 1} \|\partial_x^\alpha\partial^\beta u(t,\cdot)\|_{L^2} \le C\Big(\sum_{|\alpha|\le k,|\beta|\le 1}\|\partial_x^\alpha\partial^\beta u(0,\cdot)\|_{L^2} + \sum_{|\alpha|\le k}\sum_{i,j}\int_0^t \|\partial_x^\alpha\Box_g u\|_{L^2} + \|[\partial_x^\alpha, g_{ij}]\partial_{ij}u\|_{L^2} d\tau\Big).$$

由于 g_{ij} 的各阶导数有界, 且观察到 $[\partial_x^\alpha, g_{ij}]$ 是一个最多含有 $|\alpha|-1$ 阶空间导数的求导算子 (因为交换子会消去最高阶导数), 因而我们有

$$\sum_{|\alpha|\le k}\sum_{i,j}\|[\partial_x^\alpha, g_{ij}]\partial_{ij}u\|_{L^2} \le C\sum_{|\alpha|\le k,|\beta|\le 1}\|\partial_x^\alpha\partial^\beta u\|_{L^2}.$$

结合上面两个式子, 再在不等式两边同时对时间做积分, 应用 Gronwall 不等式, 我们可以完成我们的能量估计. □

事实上, 由于带有紧支集的光滑函数在 Sobolev 空间中稠密, 因此上面的结论可以推广到 Sobolev 空间中.

定理 2.4. *令 $0<T<\infty, k\in\mathbb{Z}_+$, 假设 g_{ij} 的各阶导数都有界. 则对于任意的 $u\in C\big([0,T],H^{k+1}\big)\cap C^1\big([0,T],H^k\big)$ 满足 $\Box_g u\in L^1\big([0,T],H^k\big)$, 我们有*

$$\sum_{|\alpha|\le 1}\|\partial^\alpha u(t,\cdot)\|_{H^k} \le C\Big(\sum_{|\alpha|\le 1}\|\partial^\alpha u(0,\cdot)\|_{H^k} + \int_0^t \|\Box_g u(\tau,\cdot)\|_{H^k} d\tau\Big).$$

其中 C 是一个和 T,k 以及 g_{ij} 的各阶导数相关的常数.

3 线性波方程解的存在性

对于经典波方程, 我们可以通过傅立叶变换等方法写出 $n\le 3$ 时解的具体表达形式, 对于更高维的情形, 可以借助分布理论写出解的表达形式, 这一部分内容感兴趣的读者可以参考 Luk 的波方程讲义.

在经典波方程中, 我们看到我们写的解还是经典意义下的解. 读者可以回顾在微分方程课程中学习的解的表达式, 很容易看出这些解都是二阶连续可微的. 对于我们考虑的波方程 (1.1), 想要写出解的具体表达形式几乎是做不到的, 甚至可以找到一些反例说明我们没法在经典意义下考虑解, 这个时候我们就需要引入之前提到的 Sobolev 空间、弱解和弱导数等概念, 我们还需要借助一些泛函分析中的抽象理论来说明解的存在性. 我们下面不加证明的叙述线性波方程解的存在性, 感兴趣的读者可以参考 [2], [4]. 线性波方程解的存在性需要借助泛函分析中的 Hahn-Banach 定理.

定理 3.1. *令 $k\in\mathbb{Z}_+, T>0$. 对于任意的 $u_0\in H^{k+1}(\mathbb{R}^n), u_1\in H^k(\mathbb{R}^n)$ 以及 $F\in L^1\big([0,T],H^k(\mathbb{R}^n)\big)$, 线性波方程 (1.1) 存在唯一的弱解*

$$u\in C^0\big([0,T],H^{k+1}\big)\cap C^1\big([0,T],H^k\big).$$

4 非线性波方程的局部适定性

在这一节中, 我们考虑方程

$$
\begin{aligned}
&-\partial_t^2 u+\sum_{i,j} g_{ij}(u,\partial u)\partial_{ij}u=F(u,\partial u),\\
&u(0,\cdot)=u_0,\quad \partial_t u(0,\cdot)=u_1.
\end{aligned}
\tag{4.1}
$$

其中 $g_{ij},F\in C^\infty,(u_0,u_1)\in H^k\times H^{k-1}$, $F(0,0)=0$ 且 (g_{ij}) 满足一致椭圆条件.

我们希望得到方程 (4.1) 的局部适定性, 即方程的解的存在唯一性和对初值的连续依赖性. 所谓的初值的连续依赖性, 用数学的语言描述即

$$u\in C^1\big([0,T],H^{k-1}\big)\cap C^0\big([0,T],H^k\big).$$

方程 (4.1) 的局部适定性已经有很多优秀的教材和讲义介绍相关的结果, 比如 [2], [4]. 在这些资料中, 我们往往需要加入 $k>n+2$ 条件来保证局部适定性的成立. 也就是说, 为了保证非线性波方程 (4.1) 是局部适定的, 我们需要对初值的 Sobolev 正则性加一些条件.

定理 4.1. *假设 $k>n+2$. 对于任意的 $(u_0,u_1)\in H^k\times H^{k-1}$, 存在一个和 $\|(u_0,u_1)\|_{H^k\times H^{k-1}}$ 有关的 $T>0$, 使得方程 (4.1) 在 $[0,T]$ 上存在唯一的弱解 $u\in C^1\big([0,T],H^{k-1}\big)\cap C^0\big([0,T],H^k\big)$.*

我们采用的证明技术是压缩映射原理, 在数学分析中我们学过

命题 4.2. *若 $f:[0,1]\to[0,1]$ 满足 $|f(x)-f(y)|\le L|x-y|$, $0<L<1$, 则必存在 $[0,1]$ 上一点 ξ_0 使得 $f(\xi_0)=\xi_0$.*

这个定理可以推广到一般完备度量空间的闭子空间. 对于度量空间等理论不熟悉的读者可以在 $[0,1]$ 的例子中培养直观感觉, 以便于理解后续证明.

证明. 记

$$A(u)=\sup_{t\in[0,T]}\big(\|u(t,\cdot)\|_{H^k}+\|\partial_t u(t,\cdot)\|_{H^{k-1}}\big),$$

我们考虑空间

$$X=\{u\in C^1\big([0,T],H^{k-1}\big)\cap C^0\big([0,T],H^k\big)|\ A(u)\le a\},$$

这里的 a,T 是待定的常数. 对于任意给定的 $u\in X$, 考虑如下线性方程

$$
\begin{aligned}
&-\partial_t^2 v+\sum_{i,j} g_{ij}(u,\partial u)\partial_{ij}v=F(u,\partial u),\\
&v(0,\cdot)=u_0,\quad \partial_t v(0,\cdot)=u_1,
\end{aligned}
$$

由线性波方程解的存在唯一性可知这样的 v 是存在唯一的, 且

$$v\in C^0\big([0,T],H^k\big)\cap C^1\big([0,T],H^{k-1}\big).$$

从而我们可以定义一个映射 ϕ

$$\phi:X\to C^0\big([0,T],H^k\big)\cap C^1\big([0,T],H^{k-1}\big).$$

下面我们说明 ϕ 是一个 X 到 X 的压缩映射. 根据压缩映射原理, 我们就可以证明解的局部适定性.

首先我们说明可以选取合适的 a 和 T, 使得 ϕ 确实把 X 映到了 X, 即说明 $A\big(\phi(u)\big) \le a$. 根据能量估计, 我们有

$$\|\phi(u)(t,\cdot)\|_{H^k} + \|\partial_t\phi(u)(t,\cdot)\|_{H^{k-1}} \le C\Big(\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}} + \int_0^t \|F(u,\partial u)(\tau,\cdot)\|_{H^{k-1}} d\tau\Big).$$

计算 $\|F(u,\partial u)\|_{H^{k-1}}$ 我们有

$$\begin{aligned}\|F(u,\partial u)\|_{H^{k-1}} &\lesssim \sum_{|\alpha|\le k-1} \|\partial_x^\alpha F(u,\partial u)\|_{L^2}\\ &\lesssim \sum_{|\alpha|\le k-1}\ \sum_{\sum_i|\beta_i|+\sum_j|\gamma_j|\le k-1} \Big\|(\partial_x^\alpha F)(u,\partial u)\partial_x^{\beta_1}u\partial_x^{\beta_2}u\cdots\partial_x^{\beta_{l_1}}u(\partial_x^{\gamma_1}\partial u)\cdots(\partial_x^{\gamma_{l_2}}\partial u)\Big\|_{L^2}.\end{aligned}$$

我们要处理的核心项就是最后一个不等号右边的式子. 由 Sobolev 嵌入定理, 我们知道

$$\|\partial_x^\beta u\|_{L^\infty} \lesssim \|\partial_x^\beta u\|_{H^{\frac{n+2}{2}}} \lesssim \|u\|_{H^{\frac{n+2}{2}+\beta}},$$

从而若 $\beta \le \frac{k+2}{2}$, 我们有

$$\|\partial_x^\beta u\|_{L^\infty} \lesssim \|u\|_{H^k}.$$

又 $F(0,0)=0, \|u\|_{H^k} + \|\partial u\|_{H^{k-1}} \le a$, 所以由上面的估计, 我们有

$$|F(u,\partial u)| \le C\Big(|u| + |\partial u|\Big) \le C\Big(\|u\|_{H^k} + \|\partial u\|_{H^{k-1}}\Big).$$

再做观察

$$\sum_{|\alpha|\le k-1}\ \sum_{\sum_i|\beta_i|+\sum_j|\gamma_j|\le k-1} \Big\|(\partial_x^\alpha F)(u,\partial u)\partial_x^{\beta_1}u\partial_x^{\beta_2}u\cdots\partial_x^{\beta_{l_1}}u(\partial_x^{\gamma_1}\partial u)\cdots(\partial_x^{\gamma_{l_2}}\partial u)\Big\|_{L^2}$$

中至多一个 β 或 $\gamma > \frac{k+2}{2}$. 回到对 $\|F(u,\partial u)\|_{H^{k-1}}$, 我们有

$$\|F(u,\partial u)\|_{H^{k-1}} \lesssim (1+\|u\|_{H^k})^k \sup_{t\in[0,T]}\Big(\|u(t,\cdot)\|_{H^k} + \|\partial_t u(t,\cdot)\|_{H^{k-1}}\Big).$$

从而我们有

$$\sup_{t\in[0,T]}\Big(\|\phi(u)(t,\cdot)\|_{H^k} + \|\partial_t\phi(u)(t,\cdot)\|_{H^{k-1}}\Big) \le C\Big(\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}} + T(1+a)^{k+1}\Big).$$

取 $a = 2C\big(\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}}\big)$, 取 T 小, 我们可以得到 $A\big(\phi(u)\big) \le a$, 从而说明 ϕ 是从 X 到 X 的映射.

然后我们来证明可以进一步选取合适的 a 和 T, 使得我们的映射 ϕ 是一个压缩映射, 即

$$|A\big(\phi(u)-\phi(v)\big)| \le L|A(u-v)|,\ 0<L<1. \tag{4.2}$$

我们让 $\phi(u)$ 和 $\phi(v)$ 的线性波方程相减, 有

$$\begin{aligned}&-\partial_t^2\big(\phi(u)-\phi(v)\big) + \sum_{i,j} g_{ij}(u,\partial u)\partial_{ij}\big(\phi(u)-\phi(v)\big)\\ &= F(u,\partial u) - F(v,\partial v) + \sum_{i,j}\Big(g_{ij}(v,\partial v) - g_{ij}(u,\partial u)\Big)\partial_{ij}\phi(v).\end{aligned}$$

注意到若我们把 $\big(\phi(u)-\phi(v)\big)$ 看成是一个整体, 则可以用类似前面证明 ϕ 将 X 映到 X 的方法来完成压缩映射部分的估计. 由于证明过程基本类似, 这部分的证明就交给感兴趣的读者了. □

注. 在证明以及后续的定理证明中我们大量用到了 $\lesssim$ 这个记号, 我们来解释它. $a \lesssim b$ 指的是存在常数 C, 使得 $a \le Cb$. 因为在上面这个定理中我们看到很多时候我们并不关心常数 C 到底是多少, 利用这个记号可以简化书写, 使得证明的关键更加清晰.

注. 在这里我们并不像之前那样需要 g_{ij} 的各阶导数有界条件 (注意, 这个条件是能量估计里要求的, 我们的能量估计依赖于这个条件). 虽然这里我们也要使用能量估计, 但是我们这个并不是用 g_{ij} 而是 $g_{ij}(u,\partial u)$, 在我们的假设之下, 利用 *Sobolev* 嵌入, 我们的 u 是属于 L^∞ 的, 从而 $g_{ij}(u,\partial u)$ 括号里的变量是一个有界量, 从而 $g_{ij}(u,\partial u)$ 自然满足各阶导数有界条件.(因为 g_{ij} 作为一个光滑函数, 它的自变量 $(u,\partial u)$ 在一个紧集上取值.)

5 进一步降低正则性

上一节中我们看到, 对于一般的非线性波方程, 我们需要条件 $k > n+2$ 来保证我们应用 Sobolev 嵌入定理的时候空间指标是合适的. 这对初值的正则性提出了比较高的要求. 对于一些具体的特殊的非线性方程, 我们可以利用方程本身的结构进一步降低方程局部适定型对正则性的要求. 在这一节里, 我们考虑方程

$$\begin{aligned} -\partial_t^2 u + \sum_i \partial_i^2 u &= u\partial_i u, \\ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) &= u_1. \end{aligned} \tag{5.1}$$

对于这样的方程, 事实上我们可以把方程的局部适定性对初值的正则性要求降低到 $k > \frac{n}{2}$. 为了完成我们的证明, 首先我们需要一个估计

引理 5.1 (Product Estimate)**.** 对任意的 $k \in \mathbb{Z}_+$, 有

$$\|fg\|_{H^k} \lesssim \|f\|_{H^k}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^k}.$$

定理 5.2. 假设 $k > \frac{n}{2}$, 对于任意的 $(u_0,u_1) \in H^k \times H^{k-1}$, 存在一个和 $\|(u_0,u_1)\|_{H^k\times H^{k-1}}$ 有关的 T, 使得方程 (5.1) 在 $[0,T]$ 上存在唯一的弱解 $u \in C^1\big([0,T],H^{k-1}\big) \cap C^0\big([0,T],H^k\big)$.

证明. 记

$$B(u) = \sup_{t\in[0,T]} \big(\|u(t,\cdot)\|_{H^k} + \|\partial_t u(t,\cdot)\|_{H^{k-1}}\big),$$

我们考虑空间

$$X = \{u \in C^1\big([0,T],H^{k-1}\big) \cap C^0\big([0,T],H^k\big) |\ B(u) \le a\}.$$

我们只证明 ϕ 是 X 到 X 的映射, 压缩映射部分的证明可以类似证明.

由能量估计, 我们有

$$\|\phi(u)(t,\cdot)\|_{H^k} + \|\partial_t\phi(u)(t,\cdot)\|_{H^{k-1}} \le C\Big(\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}} + \int_0^t \|u\partial_i u(\tau,\cdot)\|_{H^{k-1}} d\tau\Big).$$

我们利用引理 5.1 来处理 $\int_0^t \|u\partial_i u(\tau,\cdot)\|_{H^{k-1}} d\tau$,

$$\begin{aligned} \int_0^t \|u\partial_i u(\tau,\cdot)\|_{H^{k-1}} d\tau &\lesssim \int_0^t \|\partial_i(u^2)(\tau,\cdot)\|_{H^{k-1}} d\tau \\ &\lesssim \int_0^t \|u^2(\tau,\cdot)\|_{H^k} d\tau \\ &\lesssim \int_0^t \|u(\tau,\cdot)\|_{L^\infty}\|u(\tau,\cdot)\|_{H^k} d\tau \end{aligned}$$

$$\lesssim T \sup_{t\in[0,T]} \|u(t,\cdot)\|_{H^k}^2.$$

其中倒数第二个不等号用到了引理 5.1, 最后一个不等号用到了 Sobolev 嵌入定理 1.1 故我们有

$$B\big(\phi(u)\big) \le C\big(\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}} + aT\big).$$

选取 $a = 2C\Big(\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}}\Big)$ 以及 $T = \frac{1}{2C}$, 使得 ϕ 是一个从 X 到 X 的映射.

压缩映射部分的证明, 和 X 到 X 的证明完全是类似的. 感兴趣的读者可以自行尝试. 从而同样的应用不动点理论, 我们可以说明解的局部适定型. □

注. *这里笔者的 Sobolev 正则性只降到了 $\frac{n}{2}$, 事实上, 在更精细的估计下, 笔者可以将正则性要求降到 $\frac{n-1}{2}$. 囿于篇幅和工具, 这部分的内容不在这里展示, 感兴趣的读者可以联系笔者.*

注. *笔者曾经尝试证明过非线性项为 $u\partial_t u$ 的方程局部适定性对正则性的要求. 感兴趣的读者可以尝试将上面的 $u\partial_i u$ 换成 $u\partial_t u$, 试试看对正则性应该有多少要求才能做到局部适定性.*

注. *这篇文章到这里结束, 但是笔者想就笔者所学谈一谈后续的一些内容. 首先, 这篇文章只介绍了波方程的局部适定性. 所谓局部, 即我们只知道在初值时刻 $t = 0$ 附近的一个小时间区间 $[0,T]$ 内, 方程的解是存在唯一且连续依赖于初值的. 但是对于解的长时间行为, 比如解的延拓性质等这里并没有涉及. 事实上, 我们可以给出一个准则来判定局部适定性里的"局部"到底有多大, 这一部分的内容可以参见 [2] 的波方程讲义. 另一方面, 我们关心波方程的小初值整体解. 即如果初值给的足够小, 我们能否期待方程的解在 $[0,\infty)$ 上存在. 对于 $n \ge 4$ 情形, 这是可以做到的. 而对于 $n = 3$ 情形, 则需要引入一个所谓的 "null condition". 可以参见 [4]. [4] 中还介绍了波方程的向量场方法.*

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球对称时空的引力坍缩

董凯

摘 要

本文的主要目的是利用广义相对论研究球对称时空. 我们将首先对研究历史做一个简单介绍. 在此之后, 我们会引入广义相对论的一些基本概念, 以球对称时空 Einstein scalar 场为例介绍两种研究时空的常用坐标. 有了对此球对称时空的几何信息有着详细刻画之后, 我们介绍奇点与俘获面 (trapped surfaces) 的形成机制. 最后, 我们会提及在广义相对论研究中的基础, Penrose 图, 并给出一些常见时空和球对称时空中 Einstein scalar 场的 Penrose 图, 这也是 Christodoulou 在他对球对称时空研究的系列文章中主要的结果之一.

1 历史的回顾

我们知道, 广义相对论为我们提供了研究时间, 空间和引力的理论基础. 在广义相对论理论中, 时空是一个被赋予了 Lorentz 度规 (即符号为 “$-+++$” 的度规) 的 4 维流形, 其上的联络代表了时空引力. 这一理论基础便是为大家所熟知的 Einstein 方程

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{1.1}$$

其中 $R_{\mu\nu}$ 表示第 (μ,ν) 个位置的 Ricci 曲率, R 表示数量曲率, $T_{\mu\nu}$ 是能量动量应力张量, 它与外面的物质场相关.

用曲率的语言表述, Einstein 方程看起来是简洁而优美的; 但实际上, 取定局部坐标后, 这是一个极其复杂的, 由 10 个 2 阶方程组成的偏微分方程组. 在早期, 大家的研究兴趣集中于寻找它的精确解, 这就需要对时空增添一些假设.

第一个求得除了 Minkowski 时空外精确解的物理学家是 K. Schwarzchild. 他考虑的是静态球对称真空. 这时, 右边的 $T \equiv 0$, 再附加上球对称和静态消去 3 个自变量, 方程被大大简化. 通过简单的常微分方程计算, 可以得到这一解的形式是

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right), \tag{1.2}$$

其中 M 是一个常数. 可以看到, 在 $r=0$ 与 $r=2M$ 这两个点处 (当然实际是两张 3 维子流形上) 度规并非良定的, 这引发了当时数学物理学家的广泛争论. 包括后来 Kerr 得到的旋转解中的奇点现象在内, 不禁让人怀疑, 物理奇点是否是高对称性假设下的病态产物.

直到 1965 年, 著名数学物理学家 R. Penrose 发表了题为 “Gravitational Collapse and Space-time Singularities” 的著名文章. Penrose 革命性地应用了微分拓扑和黎曼几何的方法研究了时空几何并刻画它的因果结构, 并且证明: **一个闭合二维曲面上的点, 各自以不超过光速运动, 则在下一个时刻仍然会构成一个新的闭合二维曲面; 在合理的物理条件满足时, 若此二维曲面的面积单调减少, 则时空将会产生奇点!** 这给出了奇点形成的一个充分条件. 因为原定理中用到了许多与本文主旨之外的相对论中的因果概念, 故在此我们将不会给出严格的陈述和证明, 感兴趣的读者可以参考 [11]. Penrose 证明的这一结论被称为

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稿件信息: 3 月 31 日来稿, 4 月 28 日第一次修改, 5 月 1 日接受. 稿件编辑: 孙天阳.

Penrose 第一奇点定理. 它将原本难以捉摸的奇点的存在性归结于寻找时空中的一个二维俘获面 (trapped surface), 而后者是一个非常宽泛的条件. 这无疑极大推动了研究的进一步发展.

虽然已经有了奇点定理, 但我们仍未知道俘获面如何得以形成. 这是一个悬而未决的问题, 而在当时的 20 世纪 60 年代到 70 年代, 这更是一个无法克服的数学困难. 事实上, 这一问题的解决经历了足足四十年之久.

这个问题的最终解决者是著名数学家 D. Christodoulou. 20 世纪 80 年代末, 他和著名数学家 S. Klainerman 合作证明了 Minkowski 解的渐近稳定性 [6]. 这篇文章有足足 514 页, 相较于其它的波动方程论文, 不管是篇幅还是计算量都高了一个量级. 而这篇文章也几乎成为了之后 Einstein 方程研究的 “课本”. 20 世纪 90 年代末, Christodoulou 研究了球对称条件下无质量 Einstein 标量场方程, 并在此, 经过一系列文章的研究, 最终证实了 Penrose 提出的弱宇宙监督假说[1]. 最终, 在 2008 年, Christodoulou 用了共 40 年的时间, 以一己之力, 用与 Klainerman 一同建立的方法, 花了 589 页的篇幅 [5], 刻画了 Einstein 真空场方程演化中俘获面从无到有的形成过程, 为这个极其困难的数学问题的解答画上了完美的句号.

当然, 我们以上只提及了关于奇点的一部分研究历程, 而 Klainerman 与 Christodoulou 的其它众多非常著名的数学结果并未涉及 (诸如 Klainerman 以超过一千页的篇幅证明了 Einstein 场方程的局部适定性, Christodoulou 用近一千页的篇幅研究了三维相对论可压 Euler 方程的激波形成等等). 由于笔者暂时也并不懂那些部分, 也就不在此展开了.

2 广义相对论简介与问题的引入

为叙述简洁, 以下考虑的时空均指赋予了 Lorentz 度量的, 时间可连续指定的, 能全局定义的 (即良好的) 4 维流形 (M, g).

我们知道, 广义相对论将引力解释为时空的弯曲. 回顾标准力学中学习的狭义相对论, 它考虑的是不含引力 (或引力可忽略) 的情况, 也就是 Minkowski 时空 $(\mathbb{R}^4, \eta)$, 其中 $\eta = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2$, 度量中符号为负的 x^0 被解释为时间. 而在广义相对论中, 我们也有类似:

定义 2.1. *粒子在四维时空中的运动轨迹是一条曲线, 该曲线称为它的***世界线***. 对于某条世界线及其切向量 T, 若 $g(T,T) < 0$, 则称该向量是***类时向量** *(timelike vector); 若 $g(T,T) = 0$, 则称该向量是***类光向量** *(null vector); 若 $g(T,T) > 0$, 则称该向量是***类空向量** *(spacelike vector). 若某条曲线的任何一点的切向量都是类时的 (或类光, 类空的), 则称该曲线为***类时 (或类光, 类空) 曲线**.

定义 2.2. *M 的子集 Σ 称为一张***超曲面***, 若它的余维数为 1. 若一张超曲面的法向量处处类时 (或类光, 类空), 则称此超曲面为***类空 (或类光, 类时) 超曲面**.

与狭义相对论时情况相同, 我们可以简单地认为:

公理 2.1. *所有质点的世界线都是类时的, 所有光子的世界线都是类光的.*

当粒子不受除引力外的力作用时, 它的世界线是最简单的曲线. 即

命题 2.2. *自由质点的世界线是测地线.*

这样, 在广义相对论的理论框架下, 许多原本难以捉摸的问题有了很好的解释, 我们也对以前的结果有了更深的认识. 另外, 前面提到的能量动量应力张量也是广义相对论重要创

[1]我们第三小节介绍的结果是其中系列文章之一.

造. 若将它视作 4 维矩阵, 则 T_{00} 表示能量密度[2]; T_{0i} 与 T_{i0} 表示动量密度; T_{ij} 表示原先三维空间中的应力张量.

下面我们转而考虑有对称性的时空. 前面提到, 具有对称性的时空是较为简单的. 直观上解释, 球对称时空应当是具有某种球对称旋转不变性的时空. 容易理解, 对于时空 (或 Lorentz 流形) 而言, 不变性自然指的就是度规的不变性. 这指引我们定义如下 Killing 矢量场:

定义 2.3. *M 上的 Killing 向量场是指恰好生成单参数等度规微分同胚群的向量场.*

利用 Lie 导数, 容易证明

命题 2.3. *若 M 上的向量场 X 是 Killing 向量场, 则 $\mathcal{L}_X g = 0$, 或等价地, X 满足 Killing 方程:*

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0.$$

不变性得到了解释, 那么我们只需要引入一个球对称的作用, 就可以恰当地定义球对称时空. 数学上, 球对称作用自然由相应的球对称群 $SO(3)$ 给出. 这就是

定义 2.4 (球对称时空)**.** *我们称 (M, g) 是球对称的, 若其上的等度规群有一个与 $SO(3)$ 同胚的子群, 且这个子群中每个元素在 M 上的轨道要么是一个点, 要么是一个 2 维球面.*

可以想象, 在球对称假设中, 可以通过坐标选取, 消去球面上的自变量, 减少独立方程的个数. 这将在相当程度上方便对 Einstein 方程的研究.

3 两种常用坐标选取——以 Einstein scalar 场为例

设 M 是球对称时空. 这部分我们介绍研究球对称时空的两种常用坐标. 作为一个讨论的例子, 我们考虑如下的 Einstein scalar field:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}; \tag{3.1}$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi, \tag{3.2}$$

其中 ϕ 是一个函数. 通过对指标的缩并, 以上方程也等价于

$$R_{\mu\nu} = 8\pi \partial_\mu \phi \partial_\nu \phi. \tag{3.3}$$

借助 Bianchi 恒等式, 我们有守恒律:

$$\nabla^\mu T_{\mu\nu} = 0,$$

也就是

$$\Delta_g \phi = 0. \tag{3.4}$$

3.1 Einstein scalar 场来源的介绍

物理上来讲, Einstein 场方程可以由变分原理推导. 它是通过对如下的泛函求变分得到 (称为 **Einstein-Hilbert 作用量**):

$$I_G = \int \sqrt{|g|}\, \mathrm{d}x^4 \frac{1}{16\pi G}(-R + 2\Lambda),$$

[2]或质量密度. 在几何单位制下 (即 $c = G = 1$), 广义相对论将两者视为相等.

其中 Λ 为宇宙学常数. 它的变分是

$$\delta I_G = \frac{1}{16\pi G}\int \sqrt{|g|}\,\mathrm{d}x^4\left[-R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(-R+2\Lambda)\right]\delta g^{\mu\nu}.$$

若没有物质部分, 令 $\delta I_G = 0$, 利用变分原理 (Hamilton principle), 即可得到 Einstein 场方程.

若我们考虑引入物质部分的泛函为

$$I_M = \int \sqrt{|g|}\,\mathrm{d}x^4 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

其中 ϕ 是一个函数. 对 I_M 也求变分, 再令 $\delta I_G + \delta I_M = 0$, 如上利用变分原理, 即为我们的方程 (3.1) 和 (3.2)[3].

我们下面希望分别用 Bondi 坐标和 Double null 坐标来研究此时俘获面 (trapped surface) 的形成机制, 在第1节中我们就已经看到, 这直接决定了奇点的形成. 我们需要提及的是, Bondi 坐标起源于物理, 而后面的 Double null 坐标可以视作是某种意义上的 modified Bondi formalism. 读者也可以看到, 在后者坐标下计算将会变得简便许多. 事实上, Double null 坐标也是现在研究中最为广泛采用的坐标之一.

介绍两种坐标后, 我们将在最后一小节证明此时俘获面的形成机制. 较为粗略地说:

定理 3.1. *考虑球对称时空 M, 若对于一个球层的外径相较内径的增长比例比它的 Hawking 质量增长比例适当小, 则未来俘获面将在此球层中形成.*

这里的适当是指我们有一个 $O(x\log x)$ 函数控制.

3.2 Bondi 坐标

本节我们介绍 Bondi 坐标, 主要参考文献为 [4]. 最初, Bondi 坐标是由物理学家 Hermann Bondi 引入后用来研究引力波的. 他自己也曾介绍: “The 1962 paper I regard as the best scientific work I have ever done, which is later in life then mathematicians supposedly peak.”

Bondi 坐标包含了很多几何信息, 在本节我们也将看到大量几何上的计算, 这其中许多有更加一般的形式, 可以参见 [6], 当然那里的情况要复杂得多.

同 Bondi 最初构造的方法相同, 我们可以在球对称下建立 Bondi 坐标. 我们最终希望它具有形式:

$$g = -\alpha^2\mathrm{d}u^2 - \beta^2\mathrm{d}u\mathrm{d}s + r^2h_{AB}\left(\mathrm{d}x^A - U^A\mathrm{d}u\right)\left(\mathrm{d}x^B - U^B\mathrm{d}u\right), \tag{3.5}$$

其中 A 与 B 在 $1,2$ 中取值.

令 Γ 表示时空中的一条类时测地线 (对称轴), O 为 Γ 上的一点, T 为 Γ 的未来指向的单位切向量场.

我们引入函数 u, 它的梯度满足 $g(\mathrm{grad}\,u, \mathrm{grad}\,u) = 0$ (这在物理上称作 optical function), 使得它限制在 Γ 上恰度量了 Γ 的长度, 即 $u(p) = Length(\Gamma, O, p)$. 由前面定义, 我们发现 u 的水平集 $C^+(u)$ 是从 Γ 上出发的未来指向的类光测地光锥. 再定义仿射参数 s, 它满足 $(\mathrm{grad}\,u)s = -1$, 且在 Γ 上 s 取值为 0[4]. 容易验证, 这里的 u 和 s 定义合理 (即给定的限制保证了函数的存在性).

注 . *注意这里的 $\mathrm{grad}\,u$, 即 $-l$, 虽然与 u 的水平集正交, 但实际上也与之相切. 这是广义相对论 (或 Lorentz 几何) 中十分特殊的一点.*

[3]在此, 为避免赘述, 我们在此略去变分部分的复杂计算, 感兴趣的读者可以自行查阅相关资料, 如 [13].

[4]这里的 u 和 s 实际上是十分标准的函数, 它们也将成为我们之后的坐标函数.

关于此坐标函数, 我们有自然结论:

引理 3.2. *u 的水平集 $C^+(u)$ 也是球对称的. 即 Killing 矢量场中与 $SO(3)$ 同构的那个子群作用于 $C^+(u)$ 中的点后不变, 且轨道是 s 在 $C^+(u)$ 上的水平集 $S_{u,s}$, 这些水平集单点或二维球面.*

引理3.2的证明. 我们首先定义 $SO(3)$ 在时空上的作用, 它使得 Γ 恰成为对称轴 (即不动点集合). 事实上, 任意固定 $p \in \Gamma$, 设 $C^+(u,p)$ 为以它为顶点的水平集. 考虑 p 点处的切空间, 令 Σ_p 为其中与 T 正交的那个子空间. 则 Σ_p 是三维子空间, 我们可以自然定义 $SO(3)$ 在上面的作用: 任意给定 $O \in SO(3)$ 以及 $N \in \Sigma_p$, 令 $ON \in \Sigma_p$. 于是 $\forall s \geq 0$, 若 $\exp_p(sN) \in S_{u,s}$, 则 $\exp_p(sON) \in S_{u,s}$. □

下面我们定义两个坐标基矢, 它们与球面上的标准基矢共同构成时空标架. 取 $l = -\mathrm{grad}\, u$, 则 l 自然与 $S_{u,s}$ 的切空间垂直, 再取 $\underline{l}$ 为唯一的与球面 $S_{u,s}$ 切空间垂直的满足 $g(l,\underline{l}) = -2$ 的未来指向矢量.

引理 3.3. *令 Ω 为以上群作用对应的 Killing 矢量场, 则 $[\Omega, l] = [\Omega, \underline{l}] = 0$.*

引理3.3的证明. 根据引理3.2定义的群作用即知 $[\Omega, l] = 0$. 为了检验第二式, 只要分别计算 $[\Omega, \underline{l}]$ 在各个方向的分量即可. 首先是 l 方向, 由 Lie 导数运算:

$$g([\Omega, \underline{l}], l) = \Omega(g(l, \underline{l})) - g(\underline{l}, [\Omega, l]) = 0;$$

在 $\underline{l}$ 方向同上可得; 令 X 为 $S_{u,s}$ 的切向量, 则

$$g([\Omega, \underline{l}], X) = \Omega(g(X, \underline{l})) - g(\underline{l}, [\Omega, X]) = 0,$$

其中用到了 $[\Omega, X]$ 仍然为 $S_{u,s}$ 的切向量. □

我们要用到第二基本形式. 令 χ 和 $\underline{\chi}$ 分别表示 $S_{u,s}$ 相对于法向量 l 和 $\underline{l}$ 的第二基本形式, 即

$$\chi(X,Y) := g(\nabla_X l, Y), \quad \underline{\chi}(X,Y) := g(\nabla_X \underline{l}, Y).$$

引理 3.4. *χ 和 $\underline{\chi}$ 也是旋转不变的, 即*

$$\mathcal{L}_\Omega \chi = \mathcal{L}_\Omega \underline{\chi} = 0.$$

进而 $\mathrm{tr}\,\chi$ 和 $\mathrm{tr}\,\underline{\chi}$ 在二维球面 $S_{u,s}$ 上都是常数.

引理3.4的证明. 任意选取 X, Y 与 $S_{u,s}$ 相切, 我们有

$$\begin{aligned}
(\mathcal{L}_\Omega \chi)(X,Y) &= \Omega(\chi(X,Y)) - \chi(X, \mathcal{L}_\Omega Y) - \chi(\mathcal{L}_\Omega X, Y) && (\text{由 Lie 导数运算}) \\
&= [\Omega(g(\nabla_X l, Y)) - g(\nabla_X l, [\Omega, Y])] - \chi(\mathcal{L}_\Omega X, Y) && (\text{由定义}) \\
&= g([\Omega, \nabla_X l], Y) - g(\nabla_{[\Omega, X]} l, Y) && (\text{由 Lie 导数运算}) \\
&= g(\nabla_X [\Omega, l], Y) && (\text{由 } \Omega \text{ 为 Killing 场}) \\
&= 0. && (\text{由引理3.3})
\end{aligned}$$

依照类似计算, 我们也有 $\mathcal{L}_\Omega \underline{\chi} = 0$. □

为免去其它关于第二基本形式的讨论, 我们承认结果:

$$l(Area(S)) = \int_S \mathrm{tr}\,\chi \,\mathrm{dvol}_S, \quad \underline{l}(Area(S)) = \int_S \mathrm{tr}\,\underline{\chi} \,\mathrm{dvol}_S; \tag{3.6}$$

于是引入

$$h = \frac{r}{2}\operatorname{tr}\chi, \quad \underline{h} = \frac{r}{2}\operatorname{tr}\underline{\chi},$$

并利用上述引理3.4, 即可把 (3.6) 改写为

$$lr = h, \tag{3.7}$$

$$\underline{l}r = \underline{h}. \tag{3.8}$$

根据以上部分的讨论, 我们得到与前面提及的形式相同的度规

$$g = -2\mathrm{d}u\mathrm{d}s - g^{\mu\nu}\partial_\mu s\partial_\nu s\mathrm{d}u^2 + r^2\mathrm{d}\sigma,$$

其中 $\mathrm{d}\sigma$ 表示二维球面的标准度量. 对于选取的两个基矢, 也有

$$l = \frac{\partial}{\partial s}, \quad \underline{l} = 2\frac{\partial}{\partial u} - g^{\mu\nu}\partial_\mu s\partial_\nu s\frac{\partial}{\partial s}.$$

事实上, l 与 $\underline{l}$ 是与 $S_{u,s}$ 切空间垂直的, 于是它们都能写为 ∂_u 与 ∂_s 的线性组合. 由定义就有 $l = \partial_s$, 而 $g(\partial_u, \partial_s) = -g(\partial_u, \operatorname{grad} u) = -1$, 故将 $\underline{l}$ 分别与 ∂_u 与 ∂_s 作内积即得上式.

在正式推导方程之前, 我们先做一些预备工作. 定义

$$\zeta(X) := \frac{1}{2}g(\nabla_X l, \underline{l}), \quad \forall X \in TS_{u,s},$$

则与引理3.4的证明相同, 我们知道 ζ 是旋转不变的. 由于二维球面上没有旋转不变的非平凡 1-形式, 故 $\zeta = 0$. 此张量可以帮助我们得到

$$\nabla_l \underline{l} = -2\tilde{\zeta} = 0, \tag{3.9}$$

其中 $\tilde{\zeta}$ 表示与 1-形式相对应的那个向量场. 事实上, 由于

$$g(\nabla_l \underline{l}, \underline{l}) = 0, \quad g(\nabla_l \underline{l}, l) = 0,$$

可知 $\nabla_l \underline{l}$ 在 $S_{u,s}$ 的切空间内. 现在考虑, 对 $S_{u,s}$ 的切空间的任意切向量 X, 根据 $[l, X] = 0$ 将它延拓到 $C^+(u)$ 上, 则

$$2\zeta(X) = g(\nabla_X l, \underline{l}) = g(\nabla_l X, \underline{l}) = -g(\nabla_l \underline{l}, X).$$

我们再定义一个数值

$$\underline{\omega} := \frac{1}{4}g(\nabla_{\underline{l}} l, \underline{l}).$$

我们建立标架如下:

任意固定 $p \in \Gamma$, 取 e_1, e_2 为 Σ_p 对应的二维球面上的一组标准正交基, 并依照 $\nabla_l e_1 = \nabla_l e_2 = 0$ 延拓到整个以 p 为顶点的 u 的水平集上. 定义 $e_3 := l, e_4 := \underline{l}$. 则 e_1, e_2, e_3, e_4 共同构成一组零标架 (null frame), 即: 它们是一组时空标架, 且与球面正交的每个向量 (即 e_3, e_4) 都类光, 而球面上的类空向量是一组标准正交基. 下文将用 A, B 表示取值于 $1, 2$ 的抽象指标, 用 ψ 和 $\underline{\psi}$ 分别表示 $l\phi$ 和 $\underline{l}\phi$.

在计算中将用到如下熟知的 Gauss 结构定理, 证明可以参见 [12].

引理 3.5 (Gauss 定理)**.** *令 $\not{g}$ 表示 g 在 $S_{u,s}$ 上的诱导度量, K 表示 $S_{u,s}$ 的 Gauss 曲率, 则*

$$K + \frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} - \frac{1}{2}\left(\chi - \frac{1}{2}\operatorname{tr}\chi\not{g}\right)\left(\underline{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi}\not{g}\right) + \rho = 0,$$

用我们定义的几何量, 即为

$$1 + h\underline{h} = -r^2\rho,$$

其中 ρ 是一个函数, 由球对称假设, 它满足

$$R_{ABCD} = \rho\epsilon_{AB}\epsilon_{CD},$$

这里 ϵ_{AB} 表示 $S_{u,s}$ 上的体积 2-形式.

下面将涉及到大量的几何计算, 请读者不要感到厌烦, 这实际上是相关研究中必不可少的一部分. 本文中球对称情形下的计算相对简单, 我们用定义的零标架 $e_1, e_2, l, \underline{l}$ 推导方程:

1. 计算 χ_{AB} 和 $\underline{\chi}_{AB}$ 依次沿 l 和 $\underline{l}$ 的导数. 我们要用到

   引理 3.6. 我们有

   $$[\underline{l}, l] = -2\underline{\omega}l,$$

   其中 $\underline{\omega} := g(\nabla_{\underline{l}}l, \underline{l})/4$.

   引理3.6的证明. 证明是一个直接的计算, 只要注意到

   $$\nabla_l\underline{l} = -2\zeta = 0,$$

   以及

   $$g(\nabla_{\underline{l}}l, l) = 0, \quad g(\nabla_{\underline{l}}l, \underline{l}) = 4\omega,$$

   则利用标架分解就得到欲证之式. □

   以 $l\chi_{AB}$ 为例, 我们给出计算:

   $$\begin{aligned} l\chi_{AB} &= (\nabla_l\chi)(e_A, e_B) \qquad \text{(定义)} \\ &= l(\chi(e_A, e_B)) - \chi(\nabla_l e_A, e_B) - \chi(e_A, \nabla_l e_B) \\ &= \nabla_l\left(g(\nabla_{e_A}l, e_B)\right) - g(\nabla_{\nabla_l e_A}l, e_B) - g(\nabla_{\nabla_l e_B}l, e_A) \\ &= \nabla_l(e_A^\mu e_B^\nu\nabla_\mu l_\nu) - \nabla_l e_A^\mu e_B^\nu\nabla_\mu l_\nu - e_A^\mu\nabla_l e_B^\nu\nabla_\nu l_\mu \\ &= e_A^\mu e_B^\nu\nabla_l\nabla_\mu l_\nu \\ &= -e_A^\mu e_B^\nu(\nabla_\mu l^\alpha)(\nabla_\alpha l_\nu) - e_A^\mu e_B^\nu l^\alpha l^\beta R_{\mu\alpha\nu\beta}, \end{aligned}$$

   即

   $$l\chi_{AB} = -\chi_{AC}\chi_{CB} - R_{A4B4}.$$

   注意其中我们用到了 $\nabla_l l = 0$. 分别考虑上式的无迹部分和有迹部分, 由 Einstein 方程, 我们有

   $$R_{A4B4} = \frac{1}{2}R_{44}\delta_{AB} = 4\pi\psi^2\delta_{AB},$$

   以及

   $$l(\operatorname{tr}\chi) = -\frac{1}{2}(\operatorname{tr}\chi)^2 - R_{44}.$$

   这就是

   $$lh = -4\pi r\psi^2. \tag{3.10}$$

   同样道理, 我们计算其余三式并利用引理3.6, 可以得到

   $$l\underline{h} + r^{-1}h\underline{h} = -r^{-1}, \tag{3.11}$$

   $$\underline{l}h + 2\underline{\omega}h + r^{-1}h\underline{h} = -r^{-1}, \tag{3.12}$$

   $$\underline{l}\,\underline{h} - 2\underline{\omega}\,\underline{h} = -4\pi r\underline{\psi}^2. \tag{3.13}$$

2. 下面我们推导 $\underline{\omega}$ 的方程, 这个函数是由上面的方程 (3.12) 和 (3.13) 耦合进原方程的. 按照上面相同的方式可以得到

$$4l\underline{\omega} = \underline{l}^\mu \underline{l}^\nu l^\alpha \nabla_\mu \nabla_\alpha l_\nu - \underline{l}^\mu \underline{l}^\nu l^\alpha l^\beta R_{\nu\beta\mu\alpha}.$$

注意到 $\nabla_l l = 0$, 故

$$l\underline{\omega} = -\frac{1}{4} R_{3434}.$$

利用 Einstein 方程, 我们有

$$R^3_{334} = R_{34} - R^A_{3A4} = 8\pi\psi\underline{\psi},$$

故

$$\frac{1}{4} R_{3434} = 4\pi\psi\underline{\psi} - r^{-2}(1 + h\underline{h}).$$

利用引理3.5就得到

$$l\underline{\omega} = r^{-2}(1 + h\underline{h}) - 4\pi\psi\underline{\psi}. \tag{3.14}$$

3. 我们记 $\underline{a} := g^{\mu\nu}\partial_\mu s \partial_\nu s$ 为前面度规中出现过的分量, 那么我们还有 $\underline{a}$ 与 $\underline{\omega}$ 耦合的方程. 由前面部分 $\underline{l}$ 的表达式知 $\underline{a} = -\underline{l}s$, 于是

$$l\underline{a} = -(\nabla_l \underline{l}^\mu)\partial_\mu s - \underline{l}^\mu l^\nu \nabla_\mu \nabla_\nu s = \underline{l}^\mu (\nabla_\mu l^\nu)\partial_\nu s,$$

因 $\nabla_l \underline{l} = 0$ 以及 $\nabla_{\underline{l}} \nabla_l s = \underline{l}(ls) = 0$. 故上式就是

$$l\underline{a} = -2\underline{\omega}. \tag{3.15}$$

4. 这里我们推导波方程 $\Delta_g \phi = 0$ 在此坐标下的形式. 为了利用前面的几何信息, 我们自然要考虑在球方向上的投影: 令 Π 表示投影算子, 则

$$\Pi^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}(\underline{l}^\mu l^\nu + l^\mu \underline{l}^\nu),$$

在上述方程两边作用投影算子, 用 $\not{\Delta}$ 表示诱导 Laplace-Beltrami 算子, 则

$$\not{\Delta}\phi = \Pi^{\mu\nu}\nabla_\mu \nabla_\nu \phi = \frac{1}{2}(\underline{l}\psi + l\underline{\psi}) - \frac{1}{2}(\nabla_{\underline{l}} l^\nu + \nabla_l \underline{l}^\nu)\nabla_\nu \phi.$$

另一方面, 若我们考虑诱导联络 $\not{\nabla}$, 则

$$\nabla_\nu \phi = \not{\nabla}_\nu \phi - \frac{1}{2}(\underline{l}_\nu \psi + l_\nu \underline{\psi}),$$

进而

$$\not{\Delta}\phi = \Delta\phi - \frac{1}{2}(\mathrm{tr}\,\underline{\chi}\psi + \mathrm{tr}\,\chi\underline{\psi}).$$

最后两方程综合起来就是

$$\Delta\phi - \frac{1}{2}(\mathrm{tr}\,\underline{\chi}\psi + \mathrm{tr}\,\chi\underline{\psi}) = \frac{1}{2}(\underline{l}\psi + l\underline{\psi}) + \underline{\omega}\psi,$$

其中我们已经用到了引理3.6. 这就是

$$\underline{l}\psi + l\underline{\psi} + 2\underline{\omega}\psi + 2r^{-1}(\underline{h}\psi + h\underline{\psi}) = 0.$$

再利用引理3.6, 它可以进一步简化为

$$l\underline{\psi} + r^{-1}\left(h\underline{\psi} + \underline{h}\psi\right) = 0, \tag{3.16}$$

$$\underline{l}\psi + 2\underline{\omega}\psi + r^{-1}\left(h\underline{\psi} + \underline{h}\psi\right) = 0. \tag{3.17}$$

下面我们要定义一个十分重要的时空几何量, 即前面提到的 Hawking 质量:

定义 3.1. *定义二维球面 S 内的 Hawking 质量为*

$$m(S) := \frac{r(S)}{2}\left(1 + \frac{1}{16\pi}\int_S \operatorname{tr}\chi \operatorname{tr}\underline{\chi}\,\mathrm{dvol}_S\right).$$

根据以上引理3.4, 我们知道

$$m = \frac{r}{2}(1 + h\underline{h}). \tag{3.18}$$

于是用方程 (3.7), (3.8), (3.10) – (3.13), 我们有

$$lm = -2\pi r^2 \underline{h}\psi^2, \tag{3.19}$$

$$\underline{l}m = -2\pi r^2 h\underline{\psi}^2. \tag{3.20}$$

这也是我们引入 Hawking 质量的好处之一. m 不仅有很高的对称性, 且此导数的表达式也十分简洁. 这极大地方便了我们处理方程.

为了显式刻画时空中一张二维曲面的演化, 我们要介绍首先由 Penrose 给出的俘获面的定义. 像第1节提到的那样, 俘获面实际上是, 沿着每一个可能的未来方向, 面积都单调递减的那些紧致二维曲面. 可以证明, 这也等价于沿着余二维的两条类光测地线的两个方向. 于是, 在球对称下, 俘获面被定义为

定义 3.2. *球面 $S_{u,s}$ 称为俘获面, 若它满足*

$$\operatorname{tr}\chi < 0, \quad \operatorname{tr}\underline{\chi} < 0.$$

给出了方程, 我们需要计算初值条件, 关于 r 的是显然的对称性 $r|_\Gamma = 0$. 对其它量的初值:

1. h 和 $\underline{h}$ 的初值. 利用方程 (3.7), 改写为 $l(r\underline{h}) = -1$, 并考虑到如上初值, 我们有

   $$\underline{h} = -\frac{s}{r},$$

   令 $r \to 0$, 并利用标架选取可知

   $$\left(h\underline{h}\right)\big|_\Gamma = 1.$$

   考虑到 $2Tr|_\Gamma = \left(h + \underline{h}\right)\big|_\Gamma = 0$, 我们得到

   $$h|_\Gamma = 1, \quad \underline{h}\big|_\Gamma = -1. \tag{3.21}$$

2. $\underline{a}$ 的初值. 由于 $Ts = 0$, 故

   $$\underline{a}\,|_\Gamma = -\underline{l}s = ls = 1. \tag{3.22}$$

3. $\underline{\omega}$ 的初值. 利用 Γ 的测地性, 有

   $$0 = \nabla_{l+\underline{l}}\left(l + \underline{l}\right)\big|_\Gamma = -2\left(l - \underline{l}\right)\underline{\omega}.$$

   故

   $$\underline{\omega}\big|_\Gamma = 0. \tag{3.23}$$

4. ψ 和 $\underline{\psi}$ 的初值. 利用方程 (3.16), 改写为 $l(r\underline{\psi}) = -\underline{h}\psi$, 并利用如上初值, 有

   $$\underline{\psi} = -\frac{1}{r}\int_0^s \underline{h}\psi\,\mathrm{d}s. \tag{3.24}$$

注 . *在以上的方程和初值条件下, 可以证明, 方程 (3.8), (3.12) 和 (3.13) 与其它方程并不独立.*

以上的方程和初值共同构成了我们最终要考虑的方程组. 梳理一下, 如果已经在某个出射测地光锥 $C^+(u)$ 上给定了 ψ, 根据初值条件 (3.21) 和对称性, 我们可以利用方程 (3.10) 和 (3.7) 依次解出 h 和 r; 利用初值条件 (3.21) 和方程 (3.11) 可以解出 $\underline{h}$; 利用初值条件 (3.24) 得到 $\underline{\psi}$ 后, 由初值条件 (3.23) 和方程 (3.14) 解出 $\underline{\omega}$; 利用初值条件 (3.22) 和方程 (3.15) 解出 $\underline{a}$; 最后, 方程 (3.17) 可以将 ψ 的初值推向未来. 值得注意的是, 我们这里提到的时空几何量蕴含了比俘获面形成区域更广泛的信息, 这些可以最终综合为描述 Einstein scalar 场的时空的 Penrose 图. 这一点我们将在后面的章节稍作解释.

3.3 Double null 坐标

本节我们介绍 Double null 坐标, 主要参考文献为 [1]. 前面已经提到, Double null 坐标像是 Bondi 坐标的一种修正, 能使数学的讨论变得更加清晰. 总的来说, 在这时, 我们希望找到坐标系统 u, v, θ^1, θ^2, 使得度规有形式:

$$g = -4\Omega^2 \mathrm{d}u\mathrm{d}v + h_{AB}\left(\mathrm{d}\theta^A - b^A\mathrm{d}v\right)\left(\mathrm{d}\theta^B - b^B\mathrm{d}v\right). \tag{3.25}$$

从度规中就可以看到, u 与 v 作为坐标函数的水平集是类光超曲面. 此外, g 在 u 与 v 的水平集的交, 即二维曲面 $S_{u,v}$, 的诱导度量为 h_{AB}.

为得到以上坐标与相应度规, 我们当然可以同 Bondi 坐标那样对时空进行几何上的构造证明, 然后再引入许多几何量[5], 再用它们来写出 Einstein 方程的具体形式 (事实上这也是更加标准的做法). 但是因为现在已经具有了球对称假设, 那么我们就不妨 “偷个懒”, 直接用球对称的定义给出一个简化版本的 Doouble null 坐标构造.

在 M 上定义等价关系 $\sim$: $x \sim y$, 若它们在同一个 $SO(3)$ 的轨道上. 在此等价关系下, 考虑商空间 M/S, 则其中的每个点 (即每个等价类) 表示原先时空 M 当中的一张二维球面. 在不加说明的情况下, 以下计算均在 M/S 上进行.

固定时空中的一点 p, 令 Γ 代表对称轴. 在这个 2 维时空中, 我们考虑 p 点的两条类光测地线, 将其中入射 (incoming) 测地线记为 $\underline{C}$, 出射 (outgoing) 测地线记为 C. 分别记 u 与 v 为这两条测地线的仿射参数, 并在 Γ 与 $\underline{C}$ 交点处令 $u = 0$; 类似地, 沿 C 的反向, 将 C 与 Γ 交点处令 $v = 0$. 令 $\mathcal{D}(0, v_1)$ 表示 C, $\underline{C}$ 与 Γ 所围成的区域. 至此, 我们得到了一束坐标网[6], 且其中的每条坐标轴分别为 u 和 v 的函数线, 它们的水平集都是类光的 (对应到 M 上的三维超曲面上的话, 就是前面一直提到的类光超曲面). 如此函数 u 和 v 都是物理上的 optical function, 它们诱导的梯度向量场是 M 上的类光向量场, 这就是 Double null 名字的来源.

通过以上坐标的选取, 我们的度规有形式:

$$g = -\Omega^2(u,v)\mathrm{d}u\mathrm{d}v + r^2(u,v)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{3.26}$$

其中 Ω 与 r 为关于 u, v 的正函数, r 在几何上就表示球的半径.

可以看到, 在球对称假设下, 这样的构造方式更为简单, 也帮助我们避免了很多几何上的运算. 我们可以直接用上面得到的函数, 得到显式的 Einstein scalar 场方程. 首先我们计算 Christoffel 符号, 其中非零的为

$$\Gamma^0_{00} = \frac{\partial_u(\Omega^2)}{\Omega^2}, \quad \Gamma^1_{11} = \frac{\partial_v(\Omega^2)}{\Omega^2},$$
$$\Gamma^2_{02} = \Gamma^3_{03} = \frac{\partial_u r}{r},$$

[5]这里的几何量确实是 “许多” 的, 它包含 Ricci 系数: 第二基本形式 $\chi, \underline{\chi}$, 二维水平集上的 2 形式 $\eta, \underline{\eta}$(法向), $\omega, \underline{\omega}$(切向); 以及 curvature 分量: 2-形式 $\alpha, \underline{\alpha}$, 1-形式 $\beta, \underline{\beta}$, 以及函数 ρ, σ.

[6]读者可能会有疑问, 如何能保证此处的坐标网能够覆盖整个流形? 事实上, 这里只在 p 点的依赖域 (domain of dependence) 良好定义. 关于依赖域要涉及到广义相对论的因果结构问题, 那将是另一个话题, 本文在此不作详细展开.

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{\partial_v r}{r},$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \frac{1}{\tan\theta}.$$

再将它们代回曲率的表达式, 并对指标缩并成 Ricci 曲率, 我们就可以把 Einstein 场方程 (3.1) 与 (3.2) 写为:

$$r\partial_u\partial_v r + \partial_u r\partial_v r = -\frac{\Omega^2}{4}, \tag{3.27}$$
$$\partial_u\left(\frac{\partial_u r}{\Omega^2}\right) = -\frac{4\pi r(\partial_u\phi)^2}{\Omega^2}, \tag{3.28}$$
$$\partial_v\left(\frac{\partial_v r}{\Omega^2}\right) = -\frac{4\pi r(\partial_v\phi)^2}{\Omega^2}, \tag{3.29}$$
$$r\partial_u\partial_v\phi + \partial_u r\partial_v\phi + \partial_v r\partial_u\phi = 0. \tag{3.30}$$

我们当然也需要 Double null 坐标下的 Hawking 质量和俘获面的定义. 为此, 我们选取 $e_3 := 2\Omega^{-1}\partial_u$, $e_4 := 2\Omega^{-1}\partial_v$, 则

$$g(\nabla_{\partial_\theta}e_3, \partial_\theta) = \frac{2r\partial_u r}{\Omega}, \quad g(\nabla_{\partial_\theta}e_3, \partial_\phi) = 0, \quad g(\nabla_{\partial_\phi}e_3, \partial_\phi) = \frac{2r\partial_u r\sin^2\theta}{\Omega}.$$

同理计算 $\underline{\chi}$ 后, 我们得到

$$\mathrm{tr}\,\chi = \frac{4\partial_u r}{r\Omega}, \quad \mathrm{tr}\,\underline{\chi} = \frac{4\partial_v r}{r\Omega}.$$

根据前面的定义3.1和3.2, 我们定义[7]:

定义 3.3. *包含在球面 $S(u,v)$ 中的 Hawking 质量 $m(u,v)$ 定义为*

$$m(u,v) = \frac{r}{2}\left(1 + \frac{4\partial_u r\partial_v r}{\Omega^2}\right).$$

定义 3.4. *若在 (u,v) 点处满足*

$$\partial_u r(u,v) < 0, \quad \partial_v r(u,v) < 0,$$

那么我们就称 $S(u,v)$ 是球对称时空中的一张俘获面.

利用方程 (3.27), 我们发现

$$\partial_u m = -\frac{8\pi r^2\partial_v r(\partial_u\phi)^2}{\Omega^2}, \quad \partial_v m = -\frac{8\pi r^2\partial_u r(\partial_v\phi)^2}{\Omega^2}. \tag{3.31}$$

有了以上方程, 我们还需要在 Double null 坐标下提出初值条件:

(1) 显然, 我们要 $r \equiv 0$ 在 Γ 上;

(2) 通过一步正规化, 我们可以不妨设 $\Omega^2(u_0, v) \equiv 1$;

(3) 由于靠近奇点处, 将发生 "时空翻转", 即入射类光测地线转化为出射类光测地线. 这时 $\partial_u r(u_0, 0) = -\partial_v r(u_0, 0)$.

综合以上方程及初值条件, 我们可以刻画 Einstein 标量场的演化规律.

[7]细心的读者会发现这里有符号问题, 实际上这源于时空对时间定向不同.

3.4 Einstein scalar 场的时空的演化机制和奇点的形成

本节我们将重新叙述定理3.1, 并给出它的详细证明. 整体论证方式在以上两种坐标下是相同的. 限于文章篇幅, 我们将采用更为简洁的 Double null 坐进行证明, 但同时我们也会提及 Bondi 坐标下相应的结果, 对于这部分的论证, 感兴趣的读者可以自行进行.

事实上, Double null 坐标是在当前数学研究中应用最为广泛的之一. 举例来说, 在与黑洞相关的部分, 在 Schwarzchild 族黑洞解的线性稳定性和非线性稳定性, Kerr 稳定性, 真空奇点形成等等; 在与相对论联系的领域里, 它也被广泛应用, 如本文后面引用的 Einstein-Euler 理想流体下裸奇点的形成.

接上一小节的记号. 固定 C 上另一点 q 为 (u_0, v_2). 以互相包含的二维球面 p 和 q 分别作为初值, 我们考虑时空区域 $\mathcal{R}$, 这其中将出现俘获面. 其中 $\mathcal{R}$ 定义为以 pq 为一条边轴, Γ 与 C 所切割 $\underline{C}$ 得到的曲线为另一条边轴构成的 “矩形”.

我们简便记 $r_i(u) := r(u, v_i)$, $m_i(u) := m(u, v_i)$, $\Omega_i(u) := \Omega(u, v_i)$, $\phi_i(u) := \phi(u, v_i)$. 如前面定理3.1所述, 我们再定义相关半径增长比例与 Hawking 质量增长比例为:

$$\delta := \frac{r_2(u) - r_1(u)}{r_1(u)}, \quad \eta := \frac{2(m_2(u) - m_1(u))}{r_2(u)}, \quad \delta_0 := \delta(u_0), \quad \eta_0 := \eta(u_0).$$

定理3.1的精确叙述如下:

定理 3.7. *令函数*

$$E(x) := \frac{x}{(1+x)^2}\left[5 - x - \ln(2x)\right].$$

考虑初始出射测地光锥上的两张二维球面 $S_{0,1}$ 与 $S_{0,2}$, 令 $r_1(u_0)$ 与 $r_2(u_0)$ 代表它们的半径. 则对于如上的 Einstein scalar 系统, 若初值满足

$$\eta_0 > E(\delta_0),$$

则存在 u_*, 使得有俘获面形成于区域 $[u_0, u_*] \times [v_1, v_2]$, 此时 u^* 对应的出射测地光锥位于初始测地光锥的将来, 相应的二维球面 $S_{*,2}$ 达到此测地光锥的最大值, 且位于俘获域的边界上.

本定理的证明整体上是反证法. 即假定 $\partial_u r$ 及 $\partial_v r$ 其中至少有一个非负. 事实上, 我们可以断定, 这仅有可能为后者.

引理 3.8. *在 $\mathcal{D}(0, v_1)$ 及 $[u_0, 0] \times [v_1, +\infty)$ 的所有位置, 有*

$$\partial_u r \le -\frac{\Omega^2}{2} < 0.$$

或者等价地, 在 Bondi 坐标下 $\underline{h} \le -1$.

引理3.8的证明. 我们利用方程 (3.27), 考虑限制在 C 上, 则

$$\partial_v(r\partial_u r) = -\frac{1}{4},$$

于是两端从 0 到 v 积分后得到

$$r\partial_u r = -\frac{v}{4}.$$

在此式两端分别取 $v = 0$, 并利用我们的初始条件, 可见

$$\partial_v r(u_0, 0) = \frac{1}{2}, \quad \partial_u r(u_0, 0) = -\frac{1}{2}.$$

注意到方程 (3.29) 告诉我们 $\partial_v \partial_v r(u_0, v) \le 0$, 于是

$$\partial_v(r - v\partial_v r) = -v\partial_v\partial_v r \ge 0 \Longrightarrow \partial_v\left(\frac{v}{r}\right) = \frac{r - v\partial_v r}{r^2} \ge 0 \Longrightarrow \frac{v}{r} \ge \frac{1}{\partial_v r(u_0, 0)} = 2,$$

这也就是 $\partial_u r(u_0,v)\le -1/2$.

对于一般的点, 我们利用方程 (3.28), 就可以得到

$$\frac{\partial_u r}{\Omega^2}(u,v)\le\frac{\partial_u r}{\Omega^2}(u_0,v)\le-\frac{1}{2}.$$

□

由反证假设, 我们就有 $\partial_v r\ge 0$, 这也等价于 $2m\le r$.

也可以顺便得到

引理 3.9. *在同引理3.8的区域内, 我们总有 $\partial_v m\ge 0$, 进而 $m\ge 0$.*

以上两个引理也将在之后我们的估计中起到重要作用.

定理3.7的证明. 我们同样引入无量纲参数

$$x:=\frac{r_2(u)}{r_2(u_0)},$$

将 η 视作 x 的函数, 考虑 $\eta'(x)$. 同上计算:

$$\begin{aligned}\frac{\mathrm{d}\eta}{\mathrm{d}x}&=\frac{\mathrm{d}\eta}{\mathrm{d}u}\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^{-1}\\&=\frac{1}{x}\left(\frac{\partial_u(m_2-m_1)}{\partial_u r_2}-\eta\right).\end{aligned}$$

再应用前面算过的 (3.31), 我们得到

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=-\frac{\eta}{x}+\frac{16\pi}{x\partial_u r_2}\left(\frac{r_1^2\partial_v r_1(\partial_u\phi_1)^2}{\Omega_1^2}-\frac{r_2^2\partial_v r_2(\partial_u\phi_2)^2}{\Omega_2^2}\right).\tag{3.32}$$

可见, 对于上式右侧而言, 第二项是我们主要的估计目标. 为此, 我们证明

引理 3.10. *设区域 $\mathcal{D}(0,v_1)\cup\mathcal{R}$ 中没有俘获面, 则对任何 $u\in[u_0,0]$, 我们有*

$$\frac{\Omega_2^{-2}\partial_v r_2}{\Omega_1^{-2}\partial_v r_1}\le e^{-\eta}.$$

或者等价地, 在 Bondi 坐标下 $h_2h_1^{-1}\le e^{-\eta}$.

引理3.10的证明. 注意到

$$\begin{aligned}\ln\left(\Omega_2^{-2}\partial_v r_2\right)-\ln\left(\Omega_1^{-2}\partial_v r_1\right)&=\int_{v_1}^{v_2}\partial_v\ln\left(\Omega^{-2}\partial_v r\right)\mathrm{d}v\\&=-\int_{v_1}^{v_2}\frac{4\pi r(\partial_v\phi)^2}{\partial_v r}\mathrm{d}v&&\text{(由式3.29)}\\&=\int_{v_1}^{v_2}\frac{2\partial_v m}{2m-r}\mathrm{d}v&&\text{(由 Hawking 质量的定义及式3.31)}\\&\le-\int_{v_1}^{v_2}\frac{2\partial_v m}{r}\mathrm{d}v&&\text{(由引理假设)}\\&\le\frac{2(m_1-m_2)}{r_2}=-\eta.&&\text{(由引理3.9)}\end{aligned}$$

两边取幂次就是欲证结论. □

根据引理3.10, 我们继续估计得

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=-\frac{\eta}{x}+\frac{16\pi}{x\partial_u r_2}\left(\frac{r_1^2\partial_v r_1(\partial_u\phi_1)^2}{\Omega_1^2}-\frac{r_2^2\partial_v r_2(\partial_u\phi_2)^2}{\Omega_2^2}\right)$$

$$
\begin{aligned}
&= -\frac{\eta}{x} + \frac{16\pi\partial_v r_2}{x\Omega_2^2\partial_u r_2}\left(r_1^2\left(\partial_u\phi_1\right)^2\frac{\Omega_1^{-2}\partial_v r_1}{\Omega_2^{-2}\partial_v r_2} - r_2^2\left(\partial_u\phi_2\right)^2\right)\\
&\le -\frac{\eta}{x} + \frac{16\pi\partial_v r_2}{x\Omega_2^2\partial_u r_2}\left(r_1^2\left(\partial_u\phi_1\right)^2 e^{\eta} - r_2^2\left(\partial_u\phi_2\right)^2\right).
\end{aligned}
$$

下面我们处理

引理 3.11. *设区域 $\mathcal{D}(0,v_1)\cup\mathcal{R}$ 中没有俘获面. 记 $\Theta := r_2\partial_u\phi_2 - r_1\partial_u\phi_1$, 则 $\forall u\in[u_0,0]$, 我们有*

$$
\Theta^2 \le \frac{\Omega_2^2\partial_u r_2}{8\pi\partial_v r_2}(m_2-m_1)\left(\frac{1}{r_2}-\frac{1}{r_1}\right).
$$

或者等价地, 在 Bondi 坐标下, $\Theta^2 \le \underline{h}_2/(2\pi h_2)(m_2-m_1)(r_2^{-1}-r_1^{-1})$.

引理3.11的证明. 我们用方程 (3.30),

$$
\begin{aligned}
\Theta^2 &= \left(\int_{v_1}^{v_2}\partial_v(r\partial_u\phi)\mathrm{d}v\right)^2\\
&= \left(\int_{v_1}^{v_2}\partial_u r\partial_v\phi\mathrm{d}v\right)^2\\
&\le \int_{v_1}^{v_2} -\frac{8\pi r^2\partial_u r\left(\partial_u\phi\right)^2}{\Omega^2}\mathrm{d}v\int_{v_1}^{v_2} -\frac{\Omega^2\partial_u r}{8\pi r^2}\mathrm{d}v \qquad \text{(由假设及引理3.8)}\\
&= (m_2-m_1)\int_{v_1}^{v_2} -\frac{\Omega^2\partial_u r}{8\pi r^2}\mathrm{d}v.
\end{aligned}
$$

可见后面积分主要困难来自于 Ω^2 项, 此项由方程 (3.29) 耦合. 故

$$
\begin{aligned}
\int_{v_1}^{v_2} -\frac{\Omega^2\partial_u r}{r^2}\mathrm{d}v &= \int_{v_1}^{v_2}\frac{-\partial_u r\partial_v r}{r^2}\frac{\Omega^2}{\partial_v r}\mathrm{d}v\\
&\le \frac{\Omega_2^2}{\partial_v r_2}\int_{v_1}^{v_2}\frac{-\partial_u r\partial_v r}{r^2}\mathrm{d}v\\
&\le \frac{\Omega_2^2\partial_u r_2}{\partial_v r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right),
\end{aligned}
$$

其中最后一个不等号依赖于下面的断言. □

断言 3.12. *设区域 $\mathcal{D}(0,v_1)\cup\mathcal{R}$ 中没有俘获面, 则 $\partial_v\partial_u r\le 0$ 对任何的 $(u,v)\in\mathcal{R}$ 成立.*

断言的证明. 借助 Hawking 质量的定义, 我们改写方程 (3.27) 为

$$
r\partial_v\partial_u r = -\frac{m\Omega^2}{2r}.
$$

由引理3.9即得. □

根据引理3.11, 原估计式为

$$
\begin{aligned}
\frac{\mathrm{d}\eta}{\mathrm{d}x} &\le -\frac{\eta}{x} + \frac{16\pi\partial_v r_2}{x\Omega_2^2\partial_u r_2}\left(r_1^2\left(\partial_u\phi_1\right)^2 e^{2\eta} - r_2^2\left(\partial_u\phi_2\right)^2\right)\\
&= -\frac{\eta}{x} - \frac{16\pi\partial_v r_2}{x\Omega_2^2\partial_u r_2}\left(\Theta^2 + 2\Theta r_1\partial_u\phi_1 + \left(1-e^{2\eta}\right)r_1^2\left(\partial_u\phi_1\right)^2\right)\\
&\le -\frac{\eta}{x} - \frac{16\pi\partial_v r_2}{x\Omega_2^2\partial_u r_2}\left(1+\frac{1}{e^{\eta}-1}\right)\Theta^2 \qquad \text{(由 Cauchy 不等式)}\\
&\le -\frac{\eta}{x} - \frac{2}{x}(m_2-m_1)\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\left(1+\frac{1}{e^{\eta}-1}\right)
\end{aligned}
$$

$$\le -\frac{\eta}{x} + \frac{\eta\delta}{x}\left(1 + \frac{1}{e^{\eta} - 1}\right),$$

其中 $\eta \ge 0$. 由前面断言, 我们知道 $\partial_v r(u) \le \partial_v r(u_0)$, 进而 $r_2(u) - r_1(u) \le r_2(u_0) - r_1(u_0)$. 对 δ 做估计:

$$\begin{aligned}\delta(u) = \frac{r_2(u)}{r_1(u)} - 1 &= \frac{r_2(u) - r_1(u)}{r_2(u) - (r_2(u) - r_1(u))} \\ &\le \frac{r_2(u_0) - r_1(u_0)}{r_2(u) - (r_2(u_0) - r_1(u_0))} = \frac{\delta_0}{x(u)(1+\delta_0) - \delta_0}.\end{aligned}$$

利用 $e^{\eta} - 1 \ge \eta$, 我们得到

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} \le -\frac{\eta}{x} + \frac{\eta}{x}\left(1 + \frac{1}{\eta}\right)\frac{\delta_0}{x(1+\delta_0) - \delta_0}.$$

这是一个微分不等式, 利用类似 Gronwall 不等式证明方式, 可以得到

$$\eta_0 \le \eta(x)\frac{x^2}{x(1+\delta_0) - \delta_0} + \frac{\delta_0}{(1+\delta_0)^2}\left(\ln\left(\frac{1}{x(1+\delta_0) - \delta_0}\right) + \delta_0\left(\frac{1}{x(1+\delta_0) - \delta_0} - 1\right)\right).$$

我们取 $u_* \in [u_0, 0]$, 使得 $x(u_*) = 3\delta_0/(1+\delta_0)$, 此时上式右端取最小值. 将此值代入上式, 并利用 $\eta(x) < 1$ 的反证假设, 我们就得到

$$\eta_0 < E(\delta_0) = \frac{\delta_0}{(1+\delta_0)^2}\left(5 - \delta_0 - \ln(2\delta_0)\right),$$

于是我们就得到了想要的结论. □

A 附录: 球对称时空的 Penrose 图

作为附录, 我们希望介绍在广义相对论研究中起到基础作用的 Penrose 图.

如其名, Penrose 图是由著名物理学家 Penrose 创立的. 他用共形映射的方法, 可以将原本包含无限远的四维时空画在有限区域内. 由于 Penrose 图非常直观清晰, 特别是在球对称假设下可以被压缩到二维, 故在研究中经常使用. 举例来说, 用 Penrose 图可以十分简洁地解释时空的延拓, 黑洞与白洞等物理现象, 粒子的运动轨迹与奇点等.

一张 Penrose 图大致包含以下几种无穷: 类空无穷远, 过去和未来类时无穷远, 过去和未来类光无穷远; 也会包含如下几种视界: 事件视界 (event horizon), 表观视界 (apparent horizon)[8]. 其中, 对于一个黑洞, 事件视界可以简单理解为黑洞与外界的分界线: 外界粒子能通过它进入黑洞, 但再也无法从黑洞中逃逸[9], 它是一张三维类光超曲面; 而表观视界是指俘获区域 (即俘获面可能形成的区域) 的边界, 前面定理3.7中提到的 $S_{*,2}$ 实际就位于表观视界上.

我们以 Schwarzchild 时空为例解释 Penrose 图. Schwarzchild 时空即对应于球对称静态 (与时间 t 无关) 的真空 Einstein 方程的标准 Schwarzchild 解, 见 (1.2). 下面的图一即为它的 Penrose 图:

[8] 当然, 物理学家们还定义了其它的视界, 但在本文中只提及这两种.

[9] 即: 事件视界是可以与外界因果相连的区域的边界.

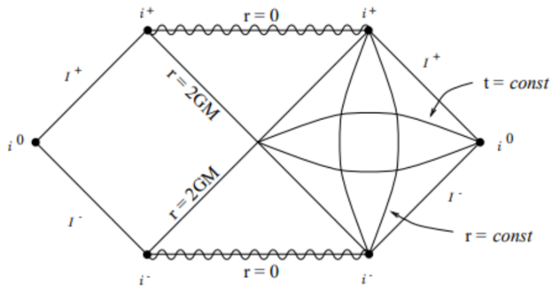

图 1: Schwarzchild 时空的 Penrose 图

在图一中, 上下两边界对应了 Schwarzchild 时空的本性奇点, 两条横线 $r = 2GM$ 为伪奇点, 也是事件视界, 是中央黑洞和白洞的边缘, 而图最外面的一周为各种无穷远. 在此图当中, t 与 r 线变成了曲线. 但是一条粒子的世界线是类时类空或类光, 只要由它在此平面中切线的倾角确定.

与前面第3节更为相关的是 Einstein scalar 场对应时空的时空图 (经过一个共形变换后即为 Penrose 图), 这也是 Christodoulou 在 [4] 中的最主要结果, 我们在此直接引用原图:

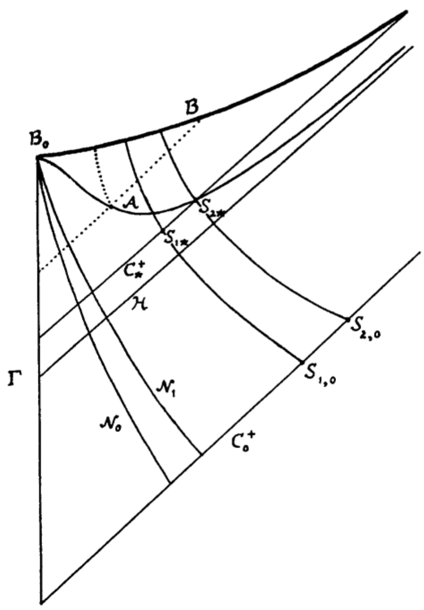

图 2: Einstein scalar 场的时空的 Penrose 图

此图也帮助解释了我们前面提到的俘获面与奇点形成机制, 我们可以直接在上面找到前面工作的区域. 如图, 左边是对称轴 Γ, 下面的 C_0^+ 为我们考虑的初值所在的出射测地光锥, 其上的 $S_{1,0}$ 与 $S_{2,0}$ 分别表示两张球面; 我们最主要的结果是找到了 $S_{2,*}$, 这是一张 C_*^+ 上的最大球面, 且它将与奇异边界 $\mathcal{B}$ 之前的俘获面形成的范围圈了起来. 这些俘获区域的边界是 $\mathcal{A}$, 即前面提到的表观视界; 而图上的 $\mathcal{H}$ 表示此时的事件视界, 它位于初值的将来, C_*^+ 的过去.

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An Introduction to Hilbert's 19th Problem

Li Xuanyu

Abstract

We discuss the existence and regularity of the minimizer of a kind of variational problem. The latter part of the paper concerning so called Hilbert's 23 problems. The basic knowledge of functional analysis, function spaces and elliptic PDEs are included. To prove the smoothness, we give the Schauder's estimate and De Giorgi's iteration.

Preface

In this paper, we would like to give a brief introduction to Hilbert's 19^{th} problem. In this problem, Hilbert questioned that whether all of the local minimizer of so called *Regular Variational Problem* is smooth. It has been solved in 1950s, 50 years after Hilbert raised it. Though has been solved for tens of years, we think it would be meaningful and illuminating to learn about how predecessors dealt with the big problems one hundred years ago.

Hilbert's problems is not as distant as it seem to be from us. In fact, the results shown in this paper is now the fundamental theories of elliptic equations. Hence this paper is written for those who are interested in the theory of PDEs and Geometry. Also, we assume that the reader only have the knowledge of mathematical analysis and linear algebra for freshmen. Therefore, all of basic knowledge is included. However, the details of proof are too much to be included in a paper no more than 20 pages. There we only give basic ideas of proof. For more details, reader can turn to reference books.

1 Preliminaries

In this section we list some basic results from Functional Analysis and PDEs, which will be used later in the paper. And in the last part in this section, we will give the definition of regular variational problem.

1.1 Preliminaries of Functional Analysis

Notation: in the following, we denote inner products by $\langle \cdot, \cdot \rangle$, norms in general space by $\| \cdot \|$, and the standard norm in $\mathbb{R}^n$ by $| \cdot |$. We always assume a vector space is over the field $\mathbb{R}$.

As learned in Linear Algebra, a vector space is a set that admits a linear structure. Functional analysis studies vector spaces that are infinite-dimensional. In this situation, algebraic structure is not enough and analytic(or topological) structure is needed.

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稿件信息: 2 月 11 日来稿, 4 月 17 日第一次修改, 5 月 2 日接受. 稿件编辑: 王子达.

We would talk about the vector spaces with inner products or norms. As learned in mathematical analysis, an inner product induces a norm, hence both of them induce a metric on a vector space. This allows us to talk about the completeness and continuity. We introduce the following terminology.

Definition 1.1. *We say a normed space is a* ***Banach space*** *if the induced metric is complete; We say a inner product space is* ***Hilbert space*** *if it is a Banach space with respect to the induced norm.*

The example of above spaces would be given in the second part this section.

Definition 1.2. *Suppose* $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ *are two normed vector space, we call a linear map* $T : X \to Y$ *a* ***linear operator*** *and its operator norm is defined as*

$$\|T\| := \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X=1} \|Tx\|_Y.$$

We say T *is* ***bounded*** *if* $\|T\| < \infty$.

The space of all bounded linear operator from X *to* Y *is denoted by* $\mathcal{L}(X, Y)$*, then* $(\mathcal{L}(X, Y), \|\cdot\|)$ *is a normed vector space. In particular, if* $Y = \mathbb{R}$*, we denote* $\mathcal{L}(X, Y)$ *by* X^* *and call it the* ***dual space*** *of* X.

The following property is basic.

Proposition 1.3. *A linear operator* T *between normed spaces* X *and* Y *is continuous if and only if it is bounded.*

Proof. If T is bounded, from $\|Tx\|_Y \leqslant \|T\|\|x\|_X$ we know T is Lipschitz-continuous. If T is unbounded, then by definition there is a sequence $\{x_n\} \subset X$ that converges to 0 but $\|Tx\|_Y \geqslant 1$. Hence T discontinuous at 0. □

Next we mainly discuss Hilbert spaces. First we list two well-known results below. Their proof can be found in [1].

Theorem 1.4. *Suppose* H *is a Hilbert space, then* $\Lambda : H \to H^*, x \mapsto \Lambda_x = \langle\cdot, x\rangle$ *is an isometric isomorphism. That is,* Λ *is bijective and preserves the norm.*

In the rest of this part we discuss the convergence in infinite-dimensional spaces. We know that in $\mathbb{R}^n$ a set is compact if and only if it is closed and bounded. However, this not hold in general case.

Proposition 1.5. *Suppose* X *is a normed vector space. Then the closed unit ball* $B := \{x \in X : \|x\| \leqslant 1\}$ *is compact if and only if* $\dim X < \infty$.

Proposition 1.5 shows that the convergence generated by norm is too strong to find converge sequences. Since compactness is important for application, we need a weaker conception of convergence.

Definition 1.6. *Suppose* H *is an inner product vector space,* $\{x_n\} \subset H$*. If for some* $x \in H$,

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle, \forall y \in H,$$

we say $\{x_n\}$ ***converges weakly*** *to* x *and* x *is the* ***weak limit*** *of* $\{x_n\}$*, denote by* $x_n \rightharpoonup x$.

By Cauchy inequality and positive definiteness of inner product, the following is obvious

Proposition 1.7. (1) *Convergence in norm induces weak convergence;* (2) *Weak limit is unique if exist.*

Now we can state our main theorem in this section, which would be used later.

Theorem 1.8. *Suppose H is a Separable Hilbert space, i.e. H has a countable dense subset S. Then any bounded sequence $\{x_n\} \subset H$ admits a weak convergence subsequence.*

Proof. Using diagonal trick and boundedness we can choose a subsequence such that $\lim_{n\to\infty}\langle x_{k_n}, y\rangle$ exists for all $y \in S$. By the denseness of S we know every $\langle x_{k_n}, y\rangle$ is a Cauchy sequence, hence the limit exists. Now $\lim_{n\to\infty}\langle x_{k_n}, y\rangle$ is a bounded functional on H, by Riesz representation theorem, there exists $x \in H$, such that x is the weak limit of $\{x_{k_n}\}$. □

At the end of this section we characterize the closedness under weak convergence.

Proposition 1.9. *Suppose H is a Hilbert space, A is a closed convex subset of H. Then A is weakly closed. That is, if $\{x_n\} \subset A$ and $x_n \rightharpoonup x$, then $x \in A$.*

Proof. Assume $x \notin A$, then $d = \operatorname{dist}(x, A) = \inf_{y\in A}\|y - x\| > 0$. Take $y_n \in A$ such that $d_n = \|y_n - x\| \to d$, then using convexity, a calculation shows

$$\|y_i - y_j\|^2 = 2(d_i^2 + d_j^2) - 4\left\|x - \frac{y_i + y_j}{2}\right\|^2 \leqslant 2\left(d_i^2 + d_j^2\right) - 4d^2 \to 0, \quad i, j \to \infty.$$

Hence $y_n \to y$ for some $y \in A$. Moreover, since A is convex, $\forall z \in A, t \in [0, 1]$, we have

$$\|y - x\|^2 \leqslant \|(1-t)y + tz - x\|^2 = \|y - x\|^2 + 2t\langle y - x, z - y\rangle + t^2\|z - y\|^2.$$

Now divided by t and let $t \to 0$ we have $\langle y - x, z - y\rangle \geqslant 0$. Take $z = y_n, z_0 = (y + x)/2$, we have

$$\langle z_0 - x, z_0 - y\rangle = -\frac{1}{4}\|y - x\|^2 < 0 \quad \text{and} \quad \langle z_0 - x_n, z_0 - y\rangle = \frac{1}{4}\|y - x\|^2 + \frac{1}{2}\langle y - x_n, x - y\rangle \geqslant 0.$$

Then let $n \to \infty$, we get a contradiction. Hence $x \in A$. □

Remark. *In fact, substitute inner product in the space with all bounded functionals, we can define weak convergence for arbitrary normed vector spaces. Then Theorem 1.5 holds for all reflexive Banach spaces, which is Eberlein-Smulyan theorem, see [1, Section 3.4] or any other functional analysis text book. And proposition 1.6 holds for all closed convex sets in normed spaces.*

1.2 Preliminaries of PDE

Notation: suppose u is a function, then Du means the gradient of u, $D_i u := \frac{\partial u}{\partial x_i}, D_{ij}u := \frac{\partial^2 u}{\partial x_i \partial x_j}$, and $D^2 u$ denote the Hessian of u. For multi-index $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{N}^n$, let $|\gamma| := \sum_{i=1}^n \gamma_i$ and

$$D^\gamma u := \frac{\partial^{|\gamma|} u}{\partial x^{\gamma_1} \cdots \partial x^{\gamma_n}}.$$

In this section and below, n always denotes the dimension of the spcae, U an open set in $\mathbb{R}^n$ and Ω a bounded open set in $\mathbb{R}^n$. For simplicity, we always assume $n \geqslant 3$.

In this section we list some basic knowledge of function spaces. For more details, see [2, 3, 4].

We begin with a characterization of continuity.

Definition 1.10. *Suppose $u : U \to \mathbb{R}$ is a function.*
(1) *If for some $C > 0, 0 < \alpha < 1$, $|u(x) - u(y)| \leqslant C|x-y|^\alpha$, we say u is **Holder continuous with exponent** α.*
(2) *$C^{k,\alpha}(\bar{U})$ denotes all the function $u \in C^k(\bar{U})$ such that all of its k^{th} derivative is Holder continuous with exponent α. If $k = 0$, we simply write $C^\alpha(\bar{U}) := C^{0,\alpha}(\bar{U})$. **Holder semi-norms** are*

$$[u]_{C^\alpha(\bar{U})} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}, [u]_{C^k(\bar{U})} := \sum_{|\gamma|=k} \sup_{\bar{U}} |D^\gamma u|, [u]_{C^{k,\alpha}(\bar{U})} := \sum_{|\gamma|=k} [D^\gamma u]_{C^\alpha(\bar{U})}.$$

The norm of $C^{k,\alpha}(\bar{U})$ is defined as

$$\|u\|_{C^{k,\alpha}(\bar{U})} := \sum_{l=0}^{k} [u]_{C^l(\bar{U})} + [u]_{C^{k,\alpha}(\bar{U})} = \sum_{|\gamma|\leqslant k} \max_{\bar{U}} |D^\gamma u| + \sum_{|\gamma|=k} \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x-y|^\alpha}.$$

Sometimes we simply write $[\cdot]_{C^\alpha(\bar{U})} = [\cdot]_{C^\alpha(U)}$, so as other norms and semi-norms. It is not difficult to check

Proposition 1.11. *$\left(C^{k,\alpha}(\bar{U}), \|\cdot\|_{C^{k,\alpha}(\bar{U})}\right)$ is a Banach space. Moreover, we have the interpolation inequalities: for $u \in C^{k,\alpha}(\bar{\Omega}), \varepsilon > 0$, there exists C_ε depends on $n, k, l, \alpha, \Omega, \varepsilon$, such that*

$$[u]_{C^l(\bar{\Omega})} \leqslant [u]_{C^{k,\alpha}(\bar{\Omega})} + C_\varepsilon \|u\|_{L^\infty(\Omega)}, 1 \leqslant l \leqslant k. \tag{I}$$

Proof. $\|\cdot\|_{C^{k,\alpha}(\bar{U})}$ is a complete norm is obvious. For interpolation inequality, we prove by contradiction. Suppose there exists $u_N \subset C^{k,\alpha}(\bar{\Omega}), \varepsilon_0 > 0$ such that $\|u_N\|_{C^k(\bar{\Omega})} = 1$ and $[u_N]_{C^l(\bar{\Omega})} > \varepsilon_0[u_N]_{C^{k,\alpha}(\bar{\Omega})} + N\|u_N\|_{L^\infty(\Omega)}$. Now, $[u_N]_{C^{k,\alpha}(\bar{\Omega})} < 1/\varepsilon_0$, by Arzela-Ascoli theorem, turn to subsequence we may assume u_N converges to some u in $C^k(\bar{\Omega})$. However, $\|u_N\|_{L^\infty(\Omega)} < 1/N$ makes $u = 0$, which contradicts $\|u_N\|_{C^k(\bar{\Omega})} = 1$. □

These interpolation inequalities are very useful. They tell us to estimate $C^{2,\alpha}$ norm, we only need to estimate $C^{2,\alpha}$ semi-norm and L^∞. Here are some useful criterion for Holder continuity. First we characterize Holder continuous by modifier.

Proposition 1.12. *Suppose ρ is a modifier on $\mathbb{R}^n$ and $u \in C^\alpha(\mathbb{R}^n)$. Let $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon), u_\varepsilon = u * \rho_\varepsilon$. Then we have the equivalence of semi-norms:*

$$\frac{1}{C}[u]_{C^\alpha(\mathbb{R}^n)} \leqslant \sup_{\varepsilon>0, x\in\mathbb{R}^n} \varepsilon^{1-\alpha}|Du_\varepsilon(x)| \leqslant C[u]_{C^\alpha(\mathbb{R}^n)},$$

where $C = C(n, \alpha, \rho) > 1$.

Proof. For the inequality on the right side, we have

$$\begin{aligned}
\varepsilon^{1-\alpha}|Du_\varepsilon(x)| &= \varepsilon^{-\alpha}\left|\int D\rho(y)\big(u(x-\varepsilon y) - u(x)\big)dy\right| \\
&\leqslant \varepsilon^{-\alpha}\int \big|D\rho(y)\big|\big|u(x-\varepsilon y) - u(x)\big|dy \\
&\leqslant [u]_{C^\alpha(\mathbb{R}^n)}\int |D\rho|,
\end{aligned}$$

where we used the the fact that ρ has compact support. And for the inequality on the left side, first we have

$$|u(x) - u(y)| \leqslant |u_\tau(x) - u(x)| + |u_\tau(x) - u_\tau(y)| + |u_\tau(y) - u(y)|.$$

Similarly,

$$|u_\tau(x)-u(x)| \leqslant \left|\int \rho(y)(u(x-\tau y)-u(x))dy\right| \leqslant \tau^\alpha [u]_{C^\alpha(\mathbb{R}^n)} \int |\rho|.$$

Also we have $|u_\tau(x)-u_\tau(y)| \leqslant \sup_{\mathbb{R}^n} |Du_\tau||x-y|$. Hence

$$|u(x)-u(y)| \leqslant C\tau^\alpha [u]_{C^\alpha(\mathbb{R}^n)} + \sup_{\mathbb{R}^n} |Du_\tau||x-y|.$$

Set $\tau = \varepsilon|x-y|, \varepsilon > 0$ to be determined. Then

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leqslant C\varepsilon^\alpha [u]_{C^\alpha(\mathbb{R}^n)} + \varepsilon^{\alpha-1}\tau^{1-\alpha} \sup_{\mathbb{R}^n} |Du_\tau| \leqslant C\varepsilon^\alpha [u]_{C^\alpha(\mathbb{R}^n)} + \varepsilon^{\alpha-1} \sup_{\tau>0, x\in\mathbb{R}^n} |Du_\tau(x)|.$$

Take ε such that $C\varepsilon^\alpha = 1/2$, we get the desired inequality. □

Proposition 1.13. *Suppose $k \in \mathbb{N} \cup \{0\}, 0 < \alpha < 1$ and $u \in L^\infty(B_1)$. If for some $M > 0$,*

$$\|u\|_{L^\infty(B_1)} \leqslant M, \left\| \frac{u(\cdot+h)-u}{|h|} \right\|_{C^{k,\alpha}\left(B_{1-|h|}\right)} \leqslant M, \forall h \in B_1.$$

Then $u \in C^{k+1,\alpha}\left(\overline{B_1}\right)$ and

$$\|u\|_{C^{k+1,\alpha}(\overline{B_1})} \leqslant C(n,k,\alpha)M.$$

Proof. It suffices to prove the case $k=0$, the general case can be easily derived from this special case.

First, the condition that $|u(x+h)-u(x)| \leqslant M|h|$ means that u is Lipschitz continuous, hence differentiable almost everywhere. Once we can show that its derivatives are Holder continuous, the u will be automatically continuously differentiable(we can show its modification will uniformly convergence itself). By assumption $v = \left(u(\cdot+h)-u\right)/|h|$ satisfies $|v(x)-v(y)| \leqslant |x-y|^\alpha, x, y \in B_{1-|h|}$ Take $y = x-h$, then

$$|u(x+h)+u(x-h)-2u(x)| \leqslant M|h|^{1+\alpha}, \forall x \in B_{1-2|h|}, h \in B_1.$$

Let $w(h) = \left(u(x+h)-u(x)\right)/|h|$, then

$$|w(h)-w(h/2)| = \frac{u(x+h)+u(x)-2u(x+h/2)}{|h|} \leqslant C(\alpha)M|h|^\alpha.$$

Thus for $k \in \mathbb{N}$, $|w(h/2^k)-w(h/2^{k+1})| \leqslant C(\alpha)M|h|^\alpha 2^{-k\alpha}$. Sum up,

$$|u(x+h)-u(x)-\langle h, Du(x)\rangle| = |h| \left|w(h) - \lim_{t\to 0} w(th)\right| \leqslant \sum_{k=0}^\infty |w(h/2^k)-w(h/2^{k+1})| \leqslant C(\alpha)M|h|^{1+\alpha}.$$

Now let $y = x+h, r = |y-x|$, choose z such that $r/2 \leqslant |z-y|, |z-x| \leqslant r$, then

$$u(y)-u(x) = \langle Du(x), y-x\rangle + O(r^{1+\alpha}),$$

$$u(z) = u(x) + \langle Du(x), z-x\rangle + O(r^{1+\alpha}) = u(y) + \langle Du(y), z-y\rangle + O(r^{1+\alpha}).$$

Take z such that $z-y$ is parallel to $Du(y)-Du(x)$, then we get $Du(y)-Du(x) = O(r^\alpha)$. □

Proposition 1.13 describes how derivatives can be approximated by difference quotient.

Same as convergence, the derivatives defined by limit is sometimes too strong to exist. It would be hard to directly show the existence of solution to PDEs with those strong derivatives. Therefore, we need a weaker conception of derivative. The space $L^1_{loc}(U)$ denotes all functions that is integrable on compact subsets of U.

Definition 1.14. *Suppose $u \in L^1_{loc}(U)$. If for some $v \in L^1_{loc}(U)$ and multi-index γ,*

$$\int_U uD^\gamma \phi = (-1)^{|\gamma|}\int_U v\phi, \forall \phi \in C_c^\infty(U)$$

*We say v is the $\gamma^{\mathbf{th}}$ **weak derivative** of u, also denotes by $v = D^\gamma u$.*

The weak derivatives is defined by integral, hence we can only distinguish them in the sense of a.e. equality. In this article, if two functions agree a.e., we will regard them as a function. By the property of integral and integrating by parts, we have

Proposition 1.15. (1) *Strong derivatives are weak derivatives;* (2) *Weak derivatives is unique if exist.*

Though looks a bit complex, weak derivatives have most of properties usual derivatives have. For example, $D^\gamma D^\psi = D^{\gamma+\psi}$ for two multi-index γ and ψ and hence commutative. Also, linearity and Leibniz's law hold. We consider the following special Sobolev space.

Definition 1.16. *Suppose $1 \leqslant p \leqslant \infty$ and k is a non-negative integer.*
(1) *The **Sobolev space** $H^k(U)$ denotes the space of all locally integrable functions u such that $D^\gamma u$ exists in the weak sense and is square integrable on U for all $|\gamma| \leqslant k$.*
(2) *The inner product of $H^k(U)$ is*

$$\langle u, v\rangle_{H^k(U)} = \sum_{|\gamma|\leqslant k}\int_U D^\gamma u D^\gamma v, \quad u, v \in H^k(U).$$

(3) *$H^k_0(U)$ denotes all functions $u \in H^k(U)$ such that $D^\gamma u$ vanishes on the ∂U for all $|\gamma| \leqslant k$.*

The basic property of Sobolev spaces is

Theorem 1.17. *For each $k \in \mathbb{N}$,$H^k(U)$ is a separable Hilbert space.*

Proof. By Minkowski's inequality, $\langle\cdot,\cdot\rangle_{H^k(U)}$ is an inner product. A direct calculation shows the completeness. Using polishing, we have $C^\infty(U) \cap H^k(U)$ is dense in $H^k(U)$. Hence polynomials is dense in $H^k(U)$, which makes $H^k(U)$ separable. □

How to study PDEs? A fundamental way is to estimate the above norms. For example, in section 3,4 we will estimate the Holder norms to show a function is the class $C^{k,\alpha}$. To establish those estimates, we need various inequalities, there we list some of them which will be used soon. Their proofs are too long to be listed here, for details, see [2, Chapter 5].

Theorem 1.18 (Sobolev's inequality)**.** *Suppose $u \in H^1_0(\Omega)$, let $2^* = \frac{2n}{n-2}$, then there exists $C = C(n, \Omega) > 0$, such that*

$$\|u\|_{L^{2^*}(\Omega)} \leqslant C\|Du\|_{L^2(\Omega)}. \tag{S}$$

Theorem 1.19 (Poincare's inequality)**.** *Suppose $u \in H^1(\Omega)$. Let $(u)_\Omega := \frac{1}{|\Omega|}\int_\Omega u$ be the average of u in Ω, then there exists $C = C(n, \Omega) > 0$, such that*

$$\|u - (u)_\Omega\|_{L^2(\Omega)} \leqslant C\|Du\|_{L^2(\Omega)}. \tag{P}$$

Theorem 1.20. *Suppose $u \in H^1(\Omega)$, then for each $\Omega' \subset\subset \Omega$ and $|h|$ small,*

$$\left\|\frac{u(\cdot+h) - u}{|h|}\right\|_{L^2(\Omega')} \leqslant \|Du\|_{L^2(\Omega)}$$

Theorem 1.21 (Rellich-Kondrachov compactness theorem)**.** *Any bounded sequence in $H^1(\Omega)$ has a subsequence that converges in $L^2(\Omega)$.*

As an application of theorem 1.21, we prove a Sobolev type inequality, which will be used later.

Proposition 1.22. *For any $\varepsilon > 0$ there exists $C_\varepsilon = C_\varepsilon(n) > 0$, such that if $u \in H^1(B_1)$ with $|\{u = 0\}| \geqslant \varepsilon|B_1|$, then*

$$\int_{B_1} u^2 \leqslant C_\varepsilon \int_{B_1} |Du|^2.$$

Proof. We prove by contradiction. Suppose there exists $\{u_k\} \subset H^1(B_1), \varepsilon_0 > 0$ such that $|\{u_k = 0\}| \geqslant \varepsilon_0|B_1|$ and $\int_{B_1} u_k^2 = 1$ but $\int_{B_1} |Du|^2 \to 0$. Then using the proposition 1.9 and theorem 1.14, turn to subsequence we may assume $u_k \to u \in H^1(B_1)$ strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$. Then $\int_{B_1} u^2 = 1$ and

$$\begin{aligned}\int_{B_1} u^2 + |Du|^2 &= \lim_{k\to\infty} \int_{B_1} uu_k + \langle Du, Du_k\rangle \\ &\leqslant \lim_{k\to\infty} \left(\int_{B_1} u^2\right)^{\frac{1}{2}} \left(\int_{B_1} u_k^2\right)^{\frac{1}{2}} + \lim_{k\to\infty} \left(\int_{B_1} |Du|^2\right)^{\frac{1}{2}} \left(\int_{B_1} |Du_k|^2\right)^{\frac{1}{2}} \\ &= 1.\end{aligned}$$

Hence $\int_{B_1} |Du|^2 = 0, Du = 0$. Then by Poincare's inequality (P), $u = (u)_{B_1}$ is a non-zero constant. However,

$$0 = \lim_{k\to\infty} \int_{B_1} |u_k - u|^2 \geqslant \liminf_{k\to\infty} \int_{\{u_k=0\}} |u_k - u|^2 \geqslant |u|^2 \inf_k |\{u_k = 0\} > 0.$$

A contradiction. □

1.3 Regular Variational Problem and Hilbert's 19^{th} Problem

Before talking about what is regular variantional problem, we need a definition.

Definition 1.23. (1) *Suppose $A : U \to \mathbb{R}^{n\times n}$ is a matrix-value function. We say A is **uniformly elliptic** if there exists $\Lambda > \lambda > 0$ such that $\lambda I \leqslant A(x) \leqslant \Lambda I$ for all $x \in \mathbb{R}^n$. That is, A is symmetry every where and*

$$\lambda|\xi|^2 \leqslant \langle A(x)\xi, \xi\rangle \leqslant \Lambda|\xi|^2, \forall x, \xi \in \mathbb{R}^n,$$

where we regard ξ as a column vector.
(2) *We say a function $L \in C^\infty(\mathbb{R}^n)$ is **uniformly convex** if D^2L is uniformly elliptic.*

We will talk about the energy functional

$$\mathcal{E}(w) := \int_\Omega L(Dw), w \in H^1(\Omega).$$

Definition 1.24. *We say $u \in H^1(\Omega)$ is a **local minimizer** of $\mathcal{E}$ if for any $\varphi \in C_c^\infty(\Omega)$ there holds*

$$\mathcal{E}(u) \leqslant \mathcal{E}(u + \varphi).$$

Remark. *$H^1(\Omega)$ is the largest place makes $\mathcal{E}$ meaningful. In fact, uniformly convexity means that L is approximately $|x|^2$. Hence to ensure $\mathcal{E}(w) < \infty$ we need at least Dw exists in the weak sense and belongs to L^2. Then do some cut-off and by Sobolev inequality, we know also $w \in L^2$, that is $w \in H^1(\Omega)$.*

Hilbert called the minimizers of $\mathcal{E}$ **regular variational problem**. What can we say about it?

Example. *If $L(x) = |x|^2$, then $\mathcal{E}$ is the energy of electronic field and the minimizer of $\mathcal{E}$ solves Laplacian equation.*

Problem. (1) *Given the boundary condition, is the global minimizer of $\mathcal{E}$ exist and unique?* (2) *Is any of local minimizer of $\mathcal{E}$ smooth?*

We will give positive answers to the above problems.

2 Existence and Uniqueness of Solution

Using the theorems listed above, we prove

Theorem 2.1. *Given any function $g : \partial\Omega \to \mathbb{R}$ that can be extended to a H^1 function on Ω. Let*

$$A := \left\{w \in H^1(\Omega) : w|_{\Omega} = g\right\}.$$

Then there exists a unique function $u \in A$, such that

$$\mathcal{E}(u) = \min_{w \in A} \mathcal{E}(w).$$

Proof. We divide the proof into 2 parts.

- Uniqueness. If $u \neq w \in A$ are two global minimizer of $\mathcal{E}$, then set $\{u \neq w\}$ has positive measure. The uniform convexity of L implies strict convexity. Hence
$$\mathcal{E}\left(\frac{u+w}{2}\right) < \frac{\mathcal{E}(u) + \mathcal{E}(w)}{2} = \mathcal{E}(u) = \mathcal{E}(w).$$
A contradiction of the minimizing property of u and w.

- Existence. Since L is convex, it has a unique minimum and $x_0 \in \mathbb{R}^n$ is such a point. By uniform convexity we have x_0 is also the minimizer of $L(x) - \frac{\lambda}{2}|x - x_0|^2$. Similarly we get
$$\frac{\lambda}{2}|x - x_0|^2 \leqslant L(x) - L(x_0) \leqslant \frac{\Lambda}{2}|x - x_0|^2.$$
Without loss of generality we assume $x_0 = 0$ and $L(0) = 0$. Now $L(x) \leqslant \frac{\Lambda}{2}|x|^2$ hence as argued before $\mathcal{E}$ is well-defined on A. Let $\mathcal{E}_0 = \min_{w \in A} \mathcal{E}(w) \geqslant 0$ and $\{u_k\} \subset A$ such that $\mathcal{E}(u_k) \to \mathcal{E}_0$. Using Sobolev inequality $\{u_k\}$ is bounded in $H^1(\Omega)$. Then by theorem1.8, proposition1.9, turn to subsequence we may assume $u_k \rightharpoonup u \in A$. To show u is a global minimizer of $\mathcal{E}$, we prove the following
Claim. *Suppose $w_k, w \in H^1(\Omega)$ such that $w_k \rightharpoonup w$, then $\mathcal{E}(w) \leqslant \liminf_{k\to\infty} \mathcal{E}(w_k)$.*
To prove the claim, for $t \in \mathbb{R}$, let $\mathcal{A}(t) = \{v \in H^1(\Omega) : \mathcal{E}(v) \leqslant t\}$, then by convexity of L, $\mathcal{A}(t)$ is also convex. Using Fatou's lemma, $\mathcal{A}(t)$ is closed in $H^1(\Omega)$. By proposition 1.9 again $\mathcal{A}(t)$ is weakly closed for all t. Now let $t^* = \liminf_{k\to\infty} \mathcal{E}(w_k)$. $\forall \epsilon > 0$, there exists a subsequence $\{w_{k_j}\}$ such that $w_{k_j} \rightharpoonup w$ and $\mathcal{E}(w_{k_j}) \leqslant t^* + \varepsilon$. Since $\{w_{k_j}\} \subset \mathcal{A}(t^* + \varepsilon)$, so does w. Let $\varepsilon \to 0$ we have proven the claim.

□

Next we derive the condition that minimizers of $\mathcal{E}$ satisfy.

Proposition 2.2. *Suppose $u \in H^1(\Omega)$ is a local minimizer of $\mathcal{E}$, then u solves $div(DL(Du)) = 0$ in the weak sense. That is*

$$\int_\Omega \langle DL(Du), D\varphi \rangle = 0, \quad \forall \varphi \in H_0^1(\Omega). \tag{*}$$

Proof. By definition, for all $\varphi \in C_c^\infty(\Omega)$, $\mathcal{E}(u + \varepsilon\varphi)$ attains its minimum at $\varepsilon = 0$. Hence $\frac{d}{d\varepsilon}\mathcal{E}(u + \varepsilon\varphi)|_{\varepsilon=0} = \int_\Omega \langle DL(Du), D\varphi \rangle = 0$. By a standard approximation, we know (*) holds for all $\phi \in H_0^1(\Omega)$. □

Remark. (*) *is called the **Euler-Lagrangian equation** of $\mathcal{E}$. It plays an important role in the proof of smoothness.*

3 Smoothness of Solution, Part I: Schauder's Estimate

3.1 Hilbert's 19^{th} Problem

Eine der begrifflich merkwürdigsten Thatsachen in den Elementen der Theorie der analytischen Funktionen erblicke ich darin, daß es Partielle Differentialgleichungen giebt, deren Integrale sämtlich notwendig analytische Funktionen der unabhängigen Variabeln sind, die also, kurz gesagt, nur analytischer Lösungen fähig sind.

——David Hilbert (1900)

David Hilbert(1862-1943), one of the greatest mathematicians in twentieth century, raised his famous 23 problems in International Congress of Mathematicians, Paris, 1900. These problems had significant influence on the development of math in twentieth century. One of them is

Hilbert's XIXth problem: suppose L is uniform convex and smooth. Is true that any local minimizer of energy functional

$$\mathcal{E}(w) = \int_\Omega L(Dw)$$

is smooth?

After Hilbert raised this question, many mathematicians devoted themselves in it and made a lot of progress. We only need to show the smoothness of solutions of (*). It was first proved that under some differentiabilities, the solution is smooth. For example, Sergei Bernstein proved that any C^3 solution of 2 variables is smooth [6]. Over the years, the requirements of differentiability needed to prove the smoothness were reduced. Finally, as a corollary of Schauder's famous estimates which published in 1934 and 1937 [7][8], we have

Theorem 3.1. *Suppose $u \in C^{1,\alpha}(\Omega)$ is a local minimizer of $\mathcal{E}$, then $u \in C^\infty(\Omega)$.*

Remark. *The initial form of Hilbert's 19^{th} consider the energy functional of the type $\mathcal{E}(w) = \int_\Omega L(Dw, w, x)$, where $L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is analytic and uniform convex. It turns out that u is also analytic. Here for simplicity we only deal with $\int_\Omega L(Dw)$.*

3.2 Schauder's Estimate

The key of the proof of theorem 3.1 is how to improve the differentiability. Informally u satisfies the equation $div\big(DL(Du)\big) = 0$. This is a non-linear PDE and hence is difficult to deal with. We translate it into a linear PDE: differentiate with regard to the i^{th} variable, we get

$$div\big(D^2L(Du)D(D_iu)\big) = 0$$

Let $v = D_iu$ and $A(x) = D^2L(Du(x))$, then we have $div(ADv) = 0$, a uniform elliptic PDE in the divergence form. This inspires us to study following equations.

Theorem 3.2 (Shauder's estimatem, non-divergence form)**.** *Suppose* $0 < \alpha < 1$*,* $a_{ij}, f \in C^\alpha(\Omega), 1 \leqslant i, j \leqslant n$ *and* $A = (a_{ij})$ *is uniform elliptic, i.e.* $0 < \lambda I \leqslant A(x) \leqslant \Lambda I$ *everywhere. If* $u \in C^{2,\alpha}(\Omega)$ *is a bounded solution of*

$$tr(A(x)D^2u(x)) = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) = f(x) \qquad in \quad \Omega.$$

Then for each $\Omega' \subset\subset \Omega$*, there holds*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leqslant C\left(n, \lambda, \Lambda, \alpha, \|a_{ij}\|_{C^\alpha(\Omega)}, \operatorname{dist}(\Omega', \partial\Omega)\right)\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)}\right).$$

There are several different proofs of theorem 3.2. We will use the method of freezing coefficient which was introduced by Trudinger. This method is to compare the solution of above PDE and the function of PDE with constant coefficient. To do this we need some basic knowledge of constant coefficient PDE.

Lemma 3.3. *Suppose* $u, f \in C^\infty(\mathbb{R}^n)$ *satisfies* $\Delta u = f$ *in* $\mathbb{R}^n$*. Then for each* $R > 0$*, we have*

$$|D_iu(x)| \leqslant \frac{n}{R}\operatorname*{osc}_{B_R(x)} u + R\sup_{B_R(x)}|f|, 1 \leqslant i \leqslant n,$$

where $\operatorname{osc}_U u := \sup_U u - \inf_U u$ *denotes the oscillation of* u *in* U*.*

Proof. This is basic for Laplacian. Integrate by parts, we get

$$\pm\frac{\partial}{\partial r}\left(r^{1-n}\int_{\partial B_r(x)} D_iudS\right) = \pm r^{1-n}\int_{B_r}\Delta D_iu = \pm r^{1-n}\int_{\partial B_r} f\nu_idS \leqslant n|B_1|\sup_{B_R(x)}|f|,$$

where ν_i denotes the i^{th} coordinate of outward pointing unit normal vector of $\partial B_R(x)$. Integrate over $(0, r]$,

$$\pm\left(r^{1-n}\int_{\partial B_r(x)} D_iudS - n|B_1|D_iu(x)\right) \leqslant n|B_1|r\sup_{B_R(x)}|f|.$$

Multiplying by r^{1-n} and again integrate over $[0, R]$, $\left|\int_{B_R(x)} D_iu - |B_1|R^nD_iu(x)\right| \leqslant |B_1|R^{n+1}\sup_{B_R(x)}|f|$, hence

$$\begin{aligned}
|D_iu(x)| &\leqslant R\sup_{B_R(x)}|f| + \frac{1}{|B_R|}\left|\int_{B_R(x)} D_iu\right| \\
&\leqslant R\sup_{B_R(x)}|f| + \frac{1}{|B_R|}\left|\int_{\partial B_R(x)}(u - u(x))\nu_idS\right| \\
&\leqslant R\sup_{B_R(x)}|f| + \frac{n}{R}\operatorname*{osc}_{B_R(x)} u.
\end{aligned}$$

□

Lemma 3.4. *Suppose* $f \in C^\alpha(\mathbb{R}^n)$ *and* $A = (a_{ij})_{n\times n}$ *is a constant matrix with* $0 < \lambda I \leqslant A \leqslant \Lambda I$*. If* $u \in C_c^{2,\alpha}(\mathbb{R}^n)$ *satisfies* $\sum_{i,j=1}^n a_{ij}D_{ij}u = f$*, then*

$$[u]_{C^{2,\alpha}(\mathbb{R}^n)} \leqslant C(n, \alpha, \lambda, \Lambda)[f]_{C^\alpha(\mathbb{R}^n)}.$$

Proof. Since the equation is $tr(AD^2u) = tr(\sqrt{A}D^2u\sqrt{A}^T) = f$, by a linear transformation we may assume $A = I$. This only change $[D^2u]_{C^\alpha(\mathbb{R}^n)}$ by a constant depend on λ, Λ. Now $\Delta u = f$, fix $B_R(x_0) \subset \mathbb{R}^n$, let $g = f - f(x_0)$, then $\Delta u - f(x_0) = g$ in $B_R(x_0)$ and $\sup_{B_R(x_0)} |g| \leqslant R^\alpha [f]_{C^\alpha(\mathbb{R}^n)}$.

We want to apply lemma 3.3. To do this, choose a modifier ρ, set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon), u_\varepsilon = u * \rho_\varepsilon, g_\varepsilon = g * \rho_\varepsilon$. Then we have $\Delta u_\varepsilon - f(x_0) = g_\varepsilon$ and $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. Moreover, for each $1 \leqslant i, j \leqslant n$ we have $\Delta D_{ij}u = D_{ij}g_\varepsilon$. Applying lemma 3.3, we get

$$\begin{aligned}|D_{ijk}u_\varepsilon(x_0)| &\leqslant \frac{n}{R}\operatorname*{osc}_{B_R(x_0)} D_{ij}u_\varepsilon + R\sup_{B_R(x_0)}|D_{ij}g_\varepsilon| \\ &\leqslant \frac{n}{R^{1-\alpha}}[D_{ij}u_\varepsilon]_{C^\alpha(B_R(x_0))} + R\sup_{B_R(x_0)}|D_{ij}g_\varepsilon|, \quad 1 \leqslant k \leqslant n.\end{aligned}$$

Recall that

$$g_\varepsilon(x) = \varepsilon^{-n}\int \rho(\frac{x-y}{\varepsilon})g(y)dy, |D_{ij}g_\varepsilon(x)| = \varepsilon^{-n-2}\left|\int D_{ij}\rho(\frac{x-y}{\varepsilon})g(y)dy\right| \leqslant C(n)\varepsilon^{-2}\sup_{B_\varepsilon(x)}|g|.$$

Hence

$$\sup_{B_R(x_0)}|D_{ij}g_\varepsilon| \leqslant C\varepsilon^{-2}\sup_{B_{R+\varepsilon}(x_0)}|g| \leqslant C\varepsilon^{-2}(R+\varepsilon)^\alpha[f]_{C^\alpha(\mathbb{R}^n)}, \varepsilon \leqslant R.$$

Write $R = N\varepsilon, N \geqslant 1$ to be determined. We get

$$\begin{aligned}\varepsilon^{1-\alpha}|D_{ijk}u_\varepsilon(x_0)| &\leqslant C\left(\left(\frac{\varepsilon}{R}\right)^{1-\alpha}[D_{ij}u]_{C^\alpha(\mathbb{R}^n)} + \frac{R(R+\varepsilon)^\alpha}{\varepsilon^{1+\alpha}}[f]_{C^\alpha(\mathbb{R}^n)}\right) \\ &= C\Big(N^{\alpha-1}[D_{ij}u]_{C^\alpha(\mathbb{R}^n)} + N(N+1)^\alpha[f]_{C^\alpha(\mathbb{R}^n)}\Big).\end{aligned}$$

Apply proposition 1.12 to $D_{ij}u$, we have

$$[D_{ij}u]_{C^\alpha(\mathbb{R}^n)} \leqslant C\sup_{\tau>0, z\in\mathbb{R}^n}\tau^{1-\alpha}|DD_{ij}u_\tau(z)| \leqslant C\Big(N^{\alpha-1}[D_{ij}u]_{C^\alpha(\mathbb{R}^n)} + N(N+1)^\alpha[f]_{C^\alpha(\mathbb{R}^n)}\Big).$$

Choose N such that $CN^{\alpha-1} = \frac{1}{2}$, then the desired estimate follows. □

With these preparations, we now begin to prove theorem 3.2.

Proof of theorem 3.2. First there exists $R_0 > 0$ such that if $R \leqslant R_0$ and $\operatorname{supp} u \subset B_R(x_0) \subset \Omega$, then

$$[u]_{C^{2,\alpha}\left(B_R(x_0)\right)} \leqslant C\left([f]_{C^\alpha\left(B_R(x_0)\right)} + \|u\|_{L^\infty\left(B_R(x_0)\right)}\right).$$

In fact, freezing coefficient, write the equation as $\sum_{i,j=1}^n a_{ij}(x_0)D_{ij}u = f - \sum_{i,j=1}^n(a_{ij} - a_{ij}(x_0))D_{ij}u$. It is easy to check $[uv]_{C^\alpha} \leqslant \|u\|_{L^\infty}[v]_{C^\alpha} + [u]_{C^\alpha}\|v\|_{L^\infty} \leqslant \|u\|_{C^\alpha}\|v\|_{C^\alpha}$, zero extend u to $\mathbb{R}^n$ and apply lemma 3.4,

$$\begin{aligned}[u]_{C^{2,\alpha}\left(B_R(x_0)\right)} &\leqslant [f - \sum_{i,j=1}^n(a_{ij} - a_{ij}(x_0))D_{ij}u]_{C^\alpha\left(B_R(x_0)\right)} \\ &\leqslant C\Big([f]_{C^\alpha\left(B_R(x_0)\right)} + R^\alpha[u]_{C^{2,\alpha}} + \|u\|_{L^\infty\left(B_R(x_0)\right)}\Big).\end{aligned}$$

Take R_0 such that $CR_0^\alpha \leqslant \frac{1}{2}, R_0 \leqslant 1$ and $R_0 \leqslant \frac{1}{2}\operatorname{dist}(\Omega', \partial\Omega)$, we get what we want. Next we remove the condition of compactly supported. For this, for each $0 < R \leqslant R_0$ and $x_0 \in \Omega'$, take a cutoff function $\zeta \in C_c^\infty\big(B_R(x_0)\big)$.

Let $v = u\zeta$, then $\sum_{i,j=1}^n a_{ij}D_{ij}v = f\zeta + u\sum_{i,j=1}^n a_{ij}D_{ij}\zeta + \sum_{i,j=1}^n a_{ij}D_iuD_j\zeta$. To apply what we have proved, we estimate the C^α semi-norm of the right-hand side. We have

$$\begin{aligned}\left[\sum_{i,j=1}^n a_{ij}D_iuD_j\zeta\right]_{C^\alpha\left(B_R(x_0)\right)} \leqslant \sum_{i,j=1}^n \|a_{ij}\|_{C^\alpha\left(B_R(x_0)\right)}\Big(&[u]_{C^1\left(B_R(x_0)\right)}[\zeta]_{C^{1,\alpha}\left(B_R(x_0)\right)}\\ &+[u]_{C^{1,\alpha}\left(B_R(x_0)\right)}[\zeta]_{C^1\left(B_R(x_0)\right)}\Big).\end{aligned}$$

Choose ζ such that $\zeta|_{B_r(x_0)} = 1, 0 \leqslant \zeta \leqslant 1$ and $[\zeta]_{C^k(B_R(x_0))} \leqslant C(R-r)^{-k}, [\zeta]_{C^{k,\alpha}(B_R(x_0))} \leqslant C(R-r)^{-(k+\alpha)}, k = 1, 2$. Using interpolation inequality I, choose ε as $\varepsilon(R-r)^{1+\alpha}, l = 1, 2, k = 2$, we get

$$\begin{aligned}\left[\sum_{i,j=1}^n a_{ij}D_iuD_j\zeta\right]_{C^\alpha\left(B_R(x_0)\right)} &\leqslant C\left(\frac{[u]_{C^1\left(B_R(x_0)\right)}}{(R-r)^{1+\alpha}} + \frac{R^{1-\alpha}[u]_{C^2\left(B_R(x_0)\right)}}{R-r}\right)\\ &\leqslant C\left(\varepsilon[u]_{C^{2,\alpha}\left(B_R(x_0)\right)} + C_\varepsilon\frac{\|u\|_{L^\infty\left(B_R(x_0)\right)}}{(R-r)^{1+\alpha}}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}[f\zeta]_{C^\alpha\left(B_R(x_0)\right)} &\leqslant \left([f]_{C^\alpha\left(B_R(x_0)\right)} + \frac{[f]_{L^\infty\left(B_R(x_0)\right)}}{(R-r)^\alpha}\right)\left[u\sum_{i,j=1}^n a_{ij}D_{ij}\zeta\right]\\ &\leqslant C\left(\varepsilon[u]_{C^\alpha\left(B_R(x_0)\right)} + C_\varepsilon\frac{\|u\|_{L^\infty\left(B_R(x_0)\right)}}{(R-r)^{2+\alpha}}\right).\end{aligned}$$

Note that $u = v$ in $B_r(x_0)$, apply what we proved, we have

$$\begin{aligned}&[u]_{C^{2,\alpha}\left(B_r(x_0)\right)} \leqslant [v]_{C^{2,\alpha}\left(B_R(x_0)\right)}\\ \leqslant& C\left([f]_{C^\alpha\left(B_{R_0}(x_0)\right)} + \varepsilon[u]_{C^{2,\alpha}\left(B_R(x_0)\right)} + \frac{\|f\|_{L^\infty\left(B_{R_0}(x_0)\right)}}{(R-r)^\alpha} + C_\varepsilon\frac{\|u\|_{L^\infty\left(B_{R_0}(x_0)\right)}}{(R-r)^{2+\alpha}}\right).\end{aligned}$$

This inequality is close to the result. To remove the $C^{2,\alpha}$ semi-norm in the right side, choose ε such that $C\varepsilon = \frac{1}{2}$, then apply following lemma to $\varphi(t) = [u]_{C^{2,\alpha}(B_t(x_0))}$, we get

$$[u]_{C^{2,\alpha}\left(B_\rho(x_0)\right)} \leqslant C\left([f]_{C^\alpha\left(B_{R_0}(x_0)\right)} + \frac{\|f\|_{L^\infty\left(B_{R_0}(x_0)\right)} + \|u\|_{L^\infty\left(B_{R_0}(x_0)\right)}}{(R-\rho)^{2+\alpha}}\right).$$

Then take $R = R_0, \rho = R_0/2$, using interpolation inequality again, we have $\|u\|_{C^{2,\alpha}\left(B_{R_0/2}(x_0)\right)} \leqslant C\left(\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\right)$, Now we conclude our proof by covering Ω' with finite many $B_{R_0/2}(x_0)$.

□

Lemma 3.5. *Suppose $\varphi : [0, R_0] \to \mathbb{R}$ is a non-negative, non-decreasing function. If there exist constants $\alpha, A, B > 0, 0 < \theta < 1$, such that*

$$\varphi(t) \leqslant \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B, 0 \leqslant t < s \leqslant R_0.$$

Then there exists $C = C(\alpha, \theta) > 0$, such that

$$\varphi(\rho) \leqslant C\left(\frac{A}{(R-\rho)^\alpha} + B\right), \forall 0 \leqslant \rho < R \leqslant R_0.$$

Proof. Take $t_0 = \rho, t_{i+1} - t_i = (1-\tau)\tau^i(R-\rho)$, $\tau > 0$ to be determined. Iterating we have

$$\varphi(t_0) \leqslant \theta\varphi(t_1) + \frac{A}{(1-\tau)^\alpha(R-\rho)^\alpha} + B \leqslant \cdots \leqslant \theta^k\varphi(t_k) + \left(\frac{A}{(1-\tau)^\alpha(R-\rho)^\alpha} + B\right)\sum_{i=0}^{k-1}\theta^i\tau^{-i\alpha}$$

Choose τ such that $\theta\tau^{-\alpha} < 1$, let $k \to \infty$. □

Another key theorem in this section is

Theorem 3.6 (Schauder's estimate, divergence form)**.** *Suppose* $0 < \alpha < 1, a_{ij} \in C^\alpha(\Omega), 1 \leqslant i, j \leqslant n$ *and* $A = (a_{ij})$ *fulfills uniformly elliptic condition* $0 < \lambda I \leqslant A(x) \leqslant \Lambda I$ *everywhere. If* $u \in C^{1,\alpha}(\Omega)$ *is bounded and*

$$\int_\Omega \langle ADu, D\varphi\rangle = 0 \quad \forall\varphi \in C_c^\infty(\Omega).$$

Then for each $\Omega' \subset\subset \Omega$*, there holds*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leqslant C\big(n, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^\alpha(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega)\big)\|u\|_{L^\infty(\Omega)}.$$

The proof of theorem is same as that of theorem 3.2, just replace lemma 3.4 by the following lemma.

Lemma 3.7. *Suppose* $f_i \in C^\alpha(\mathbb{R}^n), 1 \leqslant i \leqslant n$ *and* $A = (a_{ij})_{n\times n}$ *is a constant matrix with* $0 < \lambda I \leqslant A \leqslant \Lambda I$*. Let* $\boldsymbol{f} = (f_1, \ldots, f_n)$*, if* $u \in C_c^{1,\alpha}(\mathbb{R}^n)$ *satisfies* $\int\langle ADu, D\varphi\rangle = \int\langle A\boldsymbol{f}, D\varphi\rangle$ *for all* $\varphi \in C_c^\infty(\mathbb{R}^n)$*, then*

$$[u]_{C^{1,\alpha}(\mathbb{R}^n)} \leqslant C(n, \alpha, \lambda, \Lambda)[\boldsymbol{f}]_{C^\alpha(\mathbb{R}^n)}.$$

Proof. Set $\boldsymbol{g} = \boldsymbol{f} - \boldsymbol{f}(x_0), u_\varepsilon, \boldsymbol{g}_\varepsilon$ as in lemma 3.4, then $\mathrm{div}ADD_iu_\varepsilon = \mathrm{div}AD_i\boldsymbol{g}_\varepsilon$. Proceed as lemma 3.4. □

Combining theorem 3.2, theorem 3.6, we get:

Theorem 3.8. *Suppose* $0 < \alpha < 1, a_{ij} \in C^{k,\alpha}(\Omega), 1 \leqslant i, j \leqslant n$ *and* $A = (a_{ij})$ *fulfills uniformly elliptic condition* $0 < \lambda I \leqslant A(x) \leqslant \Lambda I$ *everywhere,* $k \in \mathbb{N}$*. If* $u \in C^{k+1,\alpha}(\Omega)$ *is a bounded solution of*

$$\int_\Omega \langle ADu, D\varphi\rangle = 0 \quad \forall\varphi \in C_c^\infty(\Omega).$$

Then for each $\Omega' \subset\subset \Omega$*, there holds*

$$\|u\|_{C^{k+1,\alpha}(\Omega')} \leqslant C\big(n, k, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^\alpha(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega)\big)\|u\|_{L^\infty(\Omega)}.$$

Proof. $k = 0$ is just theorem 3.6, then differentiate and use theorem 3.2 to do induction. Note: when $k \geqslant 1$, the equation is just $\sum_{i,j=1}^n(a_{ij}D_{ij}u + D_ia_{ij}D_ju) = 0$. □

Remark. (1) *Above Schauder's estimates are so called a prior estimate. That is, to estimate the norms, we need the initial regularity assumptions. As we will see, they also perform important role in non-linear equations;*
(2) *Theorem 3.2, 3.6 are interior estimates. If has the regularity information of boundary, one can also prove global estimates, and use them to prove the existence of solution to Dirichlet problems, see [4, Chapter 2].*

Now we can prove theorem 3.1.

Proof of theorem 3.1. Recall that $u \in C^{1,\alpha}(\Omega)$ satisfies $\int_\Omega \langle DL(Du), D\varphi\rangle = 0, \forall \varphi \in C_c^\infty(\Omega)$. Take $h \in \mathbb{R}^n$ such that $|h|$ is small, let $\Omega_\varepsilon := \{x \in \Omega | \mathrm{dist}(x, \partial\Omega) > \varepsilon\}, \varepsilon > 0$, we have

$$\int_\Omega \langle DL(Du(x+h)) - DL(Du(x)), D\varphi(x)\rangle dx = 0, \forall \varphi \in C_c^\infty(\Omega_\varepsilon).$$

Notice that $DL\big(Du(x+h)\big) - DL\big(Du(x)\big) = \int_0^1 D^2L\big(tDu(x+h) + (1-t)Du(x)\big)\big(Du(x+h) - Du(x)\big)dt$. Let $A(x) := \int_0^1 D^2L\big(tDu(x+h) + (1-t)Du(x)\big) \in C^\alpha(\Omega)$. Since L is uniformly convex, integrate we know A is uniformly elliptic. Now

$$\int_\Omega \langle A(x)D\frac{u(x+h) - u(x)}{|h|}, D\varphi\rangle = 0, \forall \varphi \in C_c^\infty(\Omega_\varepsilon).$$

Apply theorem 3.2 to $(u(x+h) - u(x))/|h|$, we have

$$\left\|\frac{u(\cdot+h) - u}{|h|}\right\|_{C^{1,\alpha}\left(B_{r/2(x_0)}\right)} \leqslant C\left\|\frac{u(\cdot+h) - u}{|h|}\right\|_{L^\infty\left(B_r(x_0)\right)} \leqslant C\|Du\|_{L^\infty(\Omega_\varepsilon)}, \forall B_r(x_0) \subset \Omega_\varepsilon.$$

The last term is independent of h, hence by proposition 1.12, we get $u \in C^{2,\alpha}(\Omega)$. Next we proceed as before by repeatedly using theorem 3.8 and proposition 1.12. We have the bootstrap argument

$$\begin{aligned} u \in C^{1,\alpha}(\Omega) &\implies A \in C^\alpha(\Omega) \implies u \in C^{2,\alpha}(\Omega) \implies \cdots \implies u \in C^{k+1,\alpha}(\Omega) \\ &\implies A \in C^{k,\alpha} \implies u \in C^{k+2,\alpha} \implies \cdots \end{aligned}$$

which lead to the conclusion $u \in C^\infty(\Omega)$. □

4 Smoothness of Solution, Part II: De Giorgi's Iteration

However, is it true that a solution u of (*) belongs H^1 implies it belongs to $C^{1,\alpha}$, was stay an open problem for many years. It was until 1956, De Giorgi(Italian, [9]) and Nash(American, [10]) independently gave positive answer to the problem. Later in 1960, Moser(German) found a different approach to the problem [11][12].

Remark. *Nash is very famous. One reason is that he contributed to many aspects of mathematics and other fields. For example, he proved Nash's embedding theorem, which claimed that any Riemann manifold can be embedded in higher dimensional Euclidean space. He also had great contribution on game theory and economics. In these fields he had his famous Nash equilibrium, for which he won the Nobel Economic Prize. But a more important reason is, the famous film* A Beautiful Mind *take him as a prototype. This film won the* 74th *Academic Award.*

The mean theorem of this section is

Theorem 4.1. *Suppose $a_{ij} \in L^\infty(\Omega), 1 \leqslant i, j \leqslant n$ and $A = (a_{ij})$ is uniformly elliptic, i.e. $0 < \lambda I \leqslant A(x) \leqslant \Lambda I$ everywhere. If $u \in H^1(\Omega)$ satisfies*

$$\int_\Omega \langle ADu, D\varphi\rangle = 0, \forall \varphi \in H_0^1(\Omega). \tag{**}$$

Then $u \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$. Moreover, for each $\Omega' \subset\subset \Omega$, there holds

$$\|u\|_{C^\alpha(\Omega')} \leqslant C(n, \frac{\Lambda}{\lambda}, \mathrm{dist}(\Omega', \partial\Omega))\|u\|_{L^2(\Omega)}.$$

Why the discovery of the theorem cost tens of years? One reason was that, although the settings seems similar to those in Schauder's estimates, but things in fact are totally different. In Schauder's estimates we deal with continuous functions. They are classical, and continuity makes the equation can be seen as perturbation of Laplacian.

However, when we turn to coefficients that are merely measurable, the thing is totally different. In this situation our equation is somewhat far away from Laplacian. The difficulty is, we need a method that is completely new.

The key is the choosing of test function φ.

Now the approach to the question from De Giorgi and Moser is called De Giorgi's iteration and Moser's iteration respectively. They have been the classical method of studying non-linear equations. Let us have a overview of them. Roughly speaking, De Giorgi's iteration is to choose φ in the form $u\eta^2$ to derive reversed Sobolev's inequality

$$\int_{\{u>k\}\cap B_r} |D(u-k)|^2 \leqslant \frac{C}{(R-r)^2}\int_{\{u>k\}\cap B_R}(u-k)^2, r<R.$$

And Moser's iteration is to choose φ in the form $u^\beta\eta^2$ to get reversed Holder's inequality

$$\left(\int_{B_r} u^{\gamma\chi}\right)^{\frac{1}{\chi}} \leqslant \frac{C}{(R-r)^2}\int_{B_R} u^\gamma, \chi>1, R>r.$$

We prove theorem 4.1 by De Giorgi's iteration. For Moser's iteration, see [4, Chapter 4] or [5, Chapter 4]. In this section, the assumption of $A=(a_{ij})$ is always as that in theorem 4.1.

Definition 4.2. *We say $u\in H^1(\Omega)$ is a* ***sub(super)solution*** *of* (**) *if for any $\varphi\in H_0^1(\Omega)$ and $\varphi\geqslant 0$, there holds*

$$\int_\Omega \langle ADu, D\varphi\rangle \leqslant (\geqslant) 0.$$

Clearly, $u\in H^1(\Omega)$ is a solution of (**) if and only if it is both the subsolution and supersolution of (**).

We divide the proof of theorem 4.1 into two parts. First, from square integrability to locally boundedness.

4.1 Local Boundedness

Theorem 4.3. *Suppose $a_{ij}\in L^\infty(\Omega), 1\leqslant i,j\leqslant n$ and $A=(a_{ij})$ is uniform elliptic, i.e. $0<\lambda I\leqslant A(x)\leqslant \Lambda I$ everywhere. If $u\in H^1(\Omega)$ is a subsolution of (**), then for each $\Omega'\subset\subset\Omega$,*

$$\sup_{\Omega'} u^+ \leqslant C(n, \frac{\Lambda}{\lambda}, \operatorname{dist}(\Omega'\partial\Omega))\|u^+\|_{L^2(\Omega)},$$

where $u^+=\max\{u,0\}$.

Proof. We only discuss the case where $\Omega=B_1$ and $\Omega'=B_{1/2}$. General case is a easy consequence of dilation and finite covering. Let $v=(u-k)^+, k\geqslant 0$, set $\varphi=v\zeta^2$, where $0\leqslant\zeta\leqslant 1$ and $\zeta\in C_c^\infty(B_1)$ to be determined. Note that $\{D\varphi\neq 0\}\subset\{u\geqslant k\}$, where $Du=Dv$, hence by uniformly elliptic condition

$$0\geqslant \int_{B_1}\langle ADu, D\varphi\rangle \geqslant \int_{B_1}\langle ADv, \zeta^2 Dv+2v\zeta D\zeta\rangle \geqslant \lambda\int_{B_1}|Dv|^2\zeta^2 - 2\Lambda\int_{B_1} v\zeta|D\zeta||Dv|.$$

Since $2v\zeta|D\zeta||Dv| \leqslant \lambda|Dv|^2\zeta^2/(2\Lambda) + 2\Lambda|D\zeta|^2v^2/\lambda$ and $|\boldsymbol{a}+\boldsymbol{b}|^2 \leqslant 2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2$, we have

$$\int_{B_1} |Dv|^2\zeta^2 \leqslant \frac{4\Lambda^2}{\lambda^2}\int_{B_1} |D\zeta|^2v^2,$$
$$\int_{B_1} |D(v\zeta)|^2 \leqslant 2\int_{B_1} v^2|D\zeta|^2 + 2\int_{B_1} |Dv|^2\zeta^2 \leqslant C\int_{B_1} |D\zeta|^2v^2.$$

Since $v\zeta \in H_0^1(B_1)$, using Sobolev's inequality (S) and Holder's inequality we have

$$\begin{aligned}\int_{B_1} (v\zeta)^2 &\leqslant \left(\int_{B_1} (v\zeta)^{2^*}\right)^{\frac{2}{2^*}} \Big|\{v\zeta \neq 0\}\Big|^{1-\frac{2}{2^*}} \leqslant C(n)\int_{B_1} |D(v\zeta)|^2\Big|\{v\zeta \neq 0\}\Big|^{\frac{2}{n}} \\ &\leqslant C\int_{B_1} |D\zeta|^2v^2\Big|\{v\zeta \neq 0\}\Big|^{\frac{2}{n}}.\end{aligned}$$

where $1/2^* = 1/2 - 1/n > 0$. Now take $0 < r < R \leqslant 1$, choose $\zeta \in C_c^\infty(B_R)$ such that $\zeta|_{B_r} = 1$ and $|D\zeta| \leqslant 2(R-r)^{-1}$, then $\{v\zeta \neq 0\} = \{x \in B_R : u(x) \geqslant k\} := A(k,R)$. Then for $0 \leqslant k < h$, $A(k,R) \supset A(h,R)$, hence

$$\int_{A(h,R)} (u-h)^2 \leqslant \int_{A(k,R)} (u-h)^2 \leqslant \int_{A(k,R)} (u-k)^2.$$

and by Chebyshev's inequality $|\{f > k\}| = \int_{\{f/k>1\}} 1 \leqslant \int_{\{f>k\}} f/k$ for $k > 0$,

$$|A(h,R)| = \Big|\{x \in B_R : u(x) - k \geqslant h - k\}\Big| \leqslant \frac{1}{(h-k)^2}\int_{A(h,R)} (u-k)^2 \leqslant \frac{1}{(h-k)^2}\int_{A(k,R)} (u-k)^2.$$

Putting these facts together, we have

$$\int_{A(h,r)} (u-h)^2 \leqslant \frac{C}{(R-r)^2}\int_{A(h,R)} (u-h)^2|A(h,R)|^{\frac{2}{n}} \leqslant \frac{C}{(h-k)^{\frac{4}{n}}(R-r)^2}\left(\int_{A(k,R)} (u-k)^2\right)^{1+\frac{2}{n}},$$

or

$$\|(u-h)^+\|_{L^2(B_r)} \leqslant \frac{C}{(h-k)^{\frac{2}{n}}(R-r)}\|(u-k)^+\|_{L^2(B_R)}^{1+\frac{2}{n}}, \forall 0 \leqslant k < h, 0 < r < R \leqslant 1.$$

Denote $\phi(k,r) := \|(u-k)^+\|_{L^2(B_r)}$, let $k_i = k(1-1/2^i), r_i = 1/2 + 1/2^{i+1}, i = 0,1,\dots$. Next we inductively choose $k > 0, \gamma > 1$ such that $\phi(k_i,r_i) \leqslant \phi(k_0,r_0)/\gamma^i$. Clearly this holds for $i = 0$. Suppose it is true for $i-1$, then

$$\begin{aligned}\phi(k_i,r_i) &\leqslant C\frac{2^{\left(\frac{2}{n}+1\right)i}}{k^{\frac{2}{n}}}\phi(k_{i-1},r_{i-1})^{1+\frac{2}{n}} \leqslant C\frac{2^{\left(1+\frac{2}{n}\right)i}}{k^{\frac{2}{n}}\gamma^{\left(1+\frac{2}{n}\right)(i-1)}}\phi(k_0,r_0)^{1+\frac{2}{n}} \\ &= C\left(\frac{2^{1+\frac{2}{n}}}{\gamma^{\frac{2}{n}}}\right)^i\left(\frac{\phi(k_0,r_0)}{k}\right)^{\frac{2}{n}}\frac{\phi(k_0,r_0)}{\gamma^i}.\end{aligned}$$

Now we choose $\gamma^{\frac{2}{n}} = 2^{1+\frac{2}{n}}$ and $\phi(k_0,r_0)/k = \|u^+\|_{L^2(B_1)}/k = C^{-\frac{n}{2}}$, then $\phi(k_i,r_i) \leqslant \phi(k_0,r_0)/\gamma^i$. Now let $i \to \infty$, we have $\phi(k,1/2) = 0$, hence

$$\sup_{B_{1/2}} u^+ \leqslant k = C'\|u^+\|_{L^2(B_1)}.$$

□

When u is a solution, u and $-u$ is both subsolution of (**), hence

Corollary 4.4. *Suppose $u \in H^1(\Omega)$ is a solution of (**), then for each $\Omega' \subset\subset \Omega$,*

$$\|u\|_{L^\infty(\Omega')} \leqslant C\Big(n, \frac{\Lambda}{\lambda}, \mathrm{dist}(\Omega', \partial\Omega)\Big)\|u\|_{L^2(\Omega)}$$

4.2 Holder Continuity

Next we show the Holder norm of a solution to (**) can be bounded by its L^∞ norm. First we need some lemmas.

Lemma 4.5. *Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex Lipschitz continuous function with $\Phi' \leqslant 0$. If $u \in H^1(\Omega)$ is a supersolution of (**) such that $\Phi(u) \in H^1(\Omega)$, then $\Phi(u)$ is a subsolution of (**).*

Proof. First assume $\Phi \in C^2(\mathbb{R})$, then $\Phi' \leqslant 0, \Phi'' \geqslant 0$. Take $\varphi \in H_0^1(\Omega)$ and $\varphi \geqslant 0$, we have

$$\int_\Omega \langle AD\Phi(u), D\varphi\rangle = \int_\Omega \Phi'(u)\langle ADu, D\varphi\rangle = -\int_\Omega \langle Du, D(-\varphi\Phi'(u)) - \int_\Omega \varphi\Phi'(u)\langle ADu, Du\rangle \leqslant 0$$

since $-\varphi\Phi'(u) \in H_0^1(\Omega)$ is non-negative and $\langle ADu, Du\rangle \geqslant 0$.

For general case, take a modifier ρ, set $\Phi_\varepsilon = \Phi * \rho_\varepsilon$, then by convexity of Φ, Φ_ε is also convex and $\Phi'_\varepsilon = \Phi' * \rho_\varepsilon \leqslant 0$. Hence $\int_\Omega \langle AD\Phi_\varepsilon(u), D\varphi\rangle \leqslant 0$. Since $\Phi_\varepsilon \to \Phi, \varepsilon \to 0$, using Lebesgue's dominant convergence theorem, we get what we want. □

Lemma 4.6. *Let $u \in H^1(B_2)$ be a positive supersolution of (**) in B_2. If $|\{u \geqslant 1\} \cap B_1| \geqslant \varepsilon|B_1|$ for some $\varepsilon > 0$, then*

$$\inf_{B_{1/2}} u \geqslant C\big(n, \varepsilon, \frac{\Lambda}{\lambda}\big) \in (0, 1).$$

Proof. Consider $u_\delta = u + \delta, \delta > 0, v = (\log u_\delta)^-$. By lemma 4.5, v is a non-negative subsolution of (**). Then theorem 4.4 implies $\sup_{B_{1/2}} v \leqslant C\|v\|_{L^2(B_1)}$. Observe that $|\{v = 0\}| = |\{u \geqslant 1\}| \geqslant \varepsilon|B_1|$, by proposition 1.22, $\|v\|_{L^2(B_1)} \leqslant C\|Dv\|_{L^2(B_1)}$. We show that $\|Dv\|_{L^2(B_1)}$ can be bounded independent of δ, then $\sup_{B_{1/2}}(\log u_\delta)^- \leqslant C$, where $C > 0$. Then $\inf_{B_{1/2}} u_\delta \geqslant e^{-C} \in (0, 1)$, just let $\delta \to 0$.

To this end, choose $\varphi = \zeta^2/u_\delta$ for some $\zeta \in C_c^\infty(B_2)$, then

$$\begin{aligned} 0 \leqslant \int_{B_2} \langle ADu_\delta\rangle, D\left(\frac{\zeta^2}{u_\delta}\right) &= 2\int_{B_2} \zeta\langle AD\log u_\delta, D\zeta\rangle - \int_{B_2} \zeta^2\langle AD\log u_\delta, D\log u_\delta\rangle \\ &\leqslant 2\Lambda \int_{B_2} \zeta|D\log u_\delta||D\zeta| - \lambda\int_{B_2} \zeta^2|D\log u_\delta|^2 \\ &\leqslant 2\Lambda \left(\int_{B_2} |D\zeta|^2\right)^{\frac{1}{2}} \left(\int_{B_2} |D\log u_\delta|^2\zeta^2\right)^{\frac{1}{2}} - \lambda\int_{B_2} \zeta^2|D\log u_\delta|^2. \end{aligned}$$

Hence $\int_{B_2} |D\log u_\delta|^2\zeta^2 \leqslant \frac{4\Lambda^2}{\lambda^2}\int_{B_2} |D\zeta|^2$. Now fix ζ with $\zeta|_{B_1} = 1$, we have the desired estimate. □

Holder continuity is a easy consequence of lemma 4.6.

proof of theorem 4.1. First we show that there exist $\gamma = \gamma(n, \lambda) \in (0, 1)$ such that if $B_{2R}(x_0) \subset \Omega$, then $\mathrm{osc}_{B_{R/2}(x_0)} u \leqslant \gamma \mathrm{osc}_{B_R(x_0)} u$.

To this end, by scaling we may assume $R = 1$, set $\alpha_1 = \sup_{B_1} u, \beta_1 = \inf_{B_1} u$. Consider two solution $u_1 := (u-\beta_1)/(\alpha_1-\beta_1)$ and $u_2 := (\alpha_1-u)/(\alpha_1-\beta_1)$, then $u \geqslant (\leqslant)\frac{1}{2}(\alpha_1+\beta_1) \iff u_1(u_2) \geqslant \frac{1}{2}$, hence there exist a u_i such that $|\{2u_i \geqslant 1\} \cap B_1| \geqslant \frac{1}{2}|B_1|$, then by lemma 4.6, this u_i satisfies $\inf_{B_{1/2}} u_i \geqslant C$ for some $C \in (0, 1)$. This equivalent to $\inf_{B_{1/2}} u \geqslant \beta_1 + C(\alpha_1 - \beta_1)$ when $i = 1$ and $\sup_{B_{1/2}} u \leqslant \alpha_1 - C(\alpha_1 - \beta_1)$ when $i = 2$. Since $\alpha_1 \geqslant \sup_{B_{1/2}} u, \beta_1 \leqslant \inf_{B_{1/2}} u$, in both cases we have $\mathrm{osc}_{B_{1/2}} u \leqslant (1 - C)\mathrm{osc}_{B_1} u$.

Then by iteration, when $B_{2R}(x_0) \subset \Omega$, $\mathrm{osc}_{B_{R/2^k}(x_0)} u \leqslant \gamma^k \mathrm{osc}_{B_R(x_0)} u$ for $k \in \mathbb{N}$. Suppose $0 < r < R$, take k such that $R/2^{k+1} < r \leqslant R/2^k$. Note that $\gamma^k = \left(\frac{1}{2}\right)^{-k\log_2\gamma} < \left(\frac{2r}{R}\right)^{-\log_2\gamma}$, set $\alpha = -\log_2\gamma$ if $\gamma > 1/2$ and $\alpha \in (0,1)$ is arbitrary if $\gamma \leqslant 1/2$. Fix $2R = \mathrm{dist}(\Omega', \partial\Omega)$, then

$$\mathrm{osc}_{B_r(x_0)} u \leqslant C r^\alpha \mathrm{osc}_{B_R(x_0)} u \leqslant C r^\alpha \|u\|_{L^\infty(\Omega'_R)} \leqslant C\|u\|_{L^2(\Omega)} r^\alpha, x_0 \in \Omega',$$

where $\Omega'_R = \{x \in \Omega | \mathrm{dist}(x, \Omega') < R\}$. Now for any $x, y \in \Omega'$ such that $|x-y| < R$, take $r = |x-y|$, then

$$|u(x) - u(y)| \leqslant \mathrm{osc}_{B_r(x)} u \leqslant C\|u\|_{L^2(\Omega)} r^\alpha = C\|u\|_{L^2(\Omega)} |x-y|^\alpha.$$

Together with corollary 4.4, we finish the proof. □

4.3 Solution to Hilbert's 19th Problem

Once we get theorem 4.1, we reach the last part of the Hilbert's 19th problem.

Theorem 4.7. *Suppose $u \in H^1(\Omega)$ is a local minimizer of $\mathcal{E}$, then $u \in C^{1,\alpha}(\Omega)$.*

Proof. The thing is basically similar to those of 3.1. Set

$$A(x) := \int_0^1 D^2 L\Big(tDu(x+h) + (1-t)Du(x)\Big),$$

by the equation (*) of u, again for $|h|$ small and $\Omega_\varepsilon = \{x \in \Omega | \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ we have

$$\int_\Omega \langle A(x) D\frac{u(x+h) - u(x)}{|h|}, D\varphi \rangle = 0, \quad \forall \varphi \in H_0^1(\Omega_\varepsilon).$$

Hence theorem 4.1 and theorem 1.20 implies

$$\left\|\frac{u(\cdot+h) - u}{|h|}\right\|_{C^\alpha\left(B_{r/2(x_0)}\right)} \leqslant C\left\|\frac{u(\cdot+h) - u}{|h|}\right\|_{L^2\left(B_r(x_0)\right)} \leqslant \|Du\|_{L^2(\Omega)}, \quad \forall B_r(x_0) \subset \Omega_\varepsilon.$$

Thanks to proposition 1.13 again, $u \in C^{1,\alpha}(\Omega)$. □

Finally, we have solved the Hilbert's problem.

Theorem. *Suppose L is uniform convex and smooth, then given any $g : \partial\Omega \to \mathbb{R}$ that can be extended to a function in $H^1(\Omega)$, there exists a unique function $u \in H^1(\Omega)$, such that $u \in C^\infty(\Omega)$ and*

$$\int_\Omega L(Du) = \inf_{\substack{w \in H^1(\Omega) \\ w|_{\partial\Omega} = g}} \int_\Omega L(Dw).$$

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主丛介绍

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摘 要

本文将从微分几何和代数几何两方面介绍主丛的基本概念与相关基本性质. 前半部分, 我们将介绍微分几何中主丛的构造和性质, 并随后尝试以直观定义联络和曲率的概念, 最后在具体的平凡丛的状况计算例子. 后半部分, 我们将介绍代数中主丛的对应——旋子的构造和性质, 并且最后介绍一种以此证明弱 Mordell-Weil 定理的方法.

考虑 M 是一个连通流形, 作为拓扑空间, 由于流形良好的性质,M 理所应当有一个万有覆叠空间 $\tilde{M}$ 和配套的覆叠映照 $\pi:\tilde{M}\to M$, 当然, 这覆叠映照 π 也是投影映射. 这不由让我们想到了纤维丛结构. 熟知的拓扑学知识告诉我们, 对连通流形 M 可以定义基本群 $\pi_1(M)$, 从而我们知道 $\forall p\in M$, 覆叠映照 π 的纤维 $\pi^{-1}(\{p\})$ 恰好同构于基本群 $\pi_1(M)$. 并且我们还熟知, 通过提升引理, 我们还可以让基本群 $\pi^{-1}(M)$ 作用到 $\tilde{M}$ 上. 此番拓扑学知识与类似的动机性的例子可以参考 [5].

于是我们获得了这样的一个结构, 这是一个流形上的纤维丛, 且每根纤维都同构于一个特定的群, 同时这个群还可以作用到这个纤维丛上. 此番结构被称之为主丛 (principal bundle), 其特殊之处尽展现于群中. 本文的后来部分, 将会具体讲述光滑的主丛,(同时群将特定为 Lie 群,) 并以一些简单的例子尝试表达一定的几何直观. 如果读者对纤维丛的基本理论不够熟悉, 可以参考 [2].

定义 0.1. *设 G 之为一个 Lie 群, 称 (E,π,M) 为一个主丛, 若 E,M 二者皆为光滑流形, 映射 $\pi:E\to M$ 是为光滑投影, 且 G 在 E 上有右作用, 且满足如下之条件:*

- *任取 $p\in M$, 纤维 $\pi^{-1}(\{p\})$ 是 G-轨道;*
- *存在 M 的开覆盖 $(U_i)_{i\in I}$, 且存在微分同胚 $\varphi_i:\pi^{-1}(U_i)\to U_i\times G$, 同时满足 G-等变.*

作为一个光滑的纤维丛, 我们希望类似向量丛那样, 介绍其联络与曲率; 而其又作为主丛这样特殊的存在, 介绍其配丛之构造. 本文实为抛砖引玉, 不过希望读者有所兴趣, 有所直观.

1 主丛的相关构造

主丛的特别之处在于其纤维是 Lie 群, 而群可以具有表示, 从而有机会将主丛变为向量丛: 这便是配丛. 为了能够理解这样的构造如何产生, 回顾一下张量积的构造也许有所裨益. 张量积的一种本质性的理解便是认为准许了 $(xa)\otimes y=x\otimes(ay)$ 这样的操作, 换言之, 将一个右作用和左作用粘贴到了一起. 现在一个主丛已经天然有了 Lie 群的右作用了, 如果 Lie 群还能左作用到另一个流形, 我们可以相信, 我们也可以构造一个新的纤维丛.

考虑一个流形 N 有 G-左作用,(注意到当 N 恰是欧氏空间时, 我们把这样的作用称为表示,) 我们可以定义 $E_N=(E\times N)/G$, 其中商掉的群作用是 $(x,u)g=(xg,g^{-1}u)$. 最后仍需要定义投影映射, 为此, 我们可以作出如下交换图:

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稿件信息: 3 月 25 日来稿, 5 月 10 日第一次修改, 5 月 10 日接受. 稿件编辑: 孙天阳.

$$
\begin{tikzcd}
E\times N \arrow[r, "\pi_1"] \arrow[d] & E \arrow[r, "\pi"] & M \\
(E\times N)/G \arrow[urr, dashed, "\pi_N"] & &
\end{tikzcd}
$$

我们可以简单来感受一下该纤维丛的局部平凡化是如何得到的, 实际上只需要有

$$\pi_N^{-1}(U) \simeq (\pi^{-1}(U)\times N)/G \simeq ((U\times G)\times N)/G \simeq U\times N.$$

至此, 我们得到了 (E_N, π_N, M), 被称为相伴于主丛 (E, π, M) 的以 N 为纤维的纤维丛. 当 N 正是线性空间时, 作用即为表示, 这恰好就是向量丛, 这也是主丛一种常见的用法: 即通过主丛制造向量丛. 一大优点在于, 相比逐纤维构造然后无交并, 我们可以不用担心许多的光滑性问题. 在物理的许多具体问题中, 这一方法更有其独到之处.

有了一个结构, 结构之间的同态显然是必然要构造与定义的. 对于主丛, 最关键的便在于 Lie 群 G 给出的右作用, 在定义同态的时候也需要明显展出, 是为:

定义 1.1. *(E, π, M) 和 (E', π', M) 为两个 G-主丛, 则主丛间的同态 $\varphi : E \to E'$ 首先是纤维丛之间的同态, 并且满足 G-等变.*

有了同态, 同构的定义自然呼之欲出, 此处不书. 实际上我们同样可以定义底流形变动的状况下的同态, 此时应该对映射对去定义, 如下:

定义 1.2. *映射对 $(f, \bar{f})$ 为两个 G-主丛 (E, π, M) 和 (E', π', M') 之间的同态, 若 $f : M \to M'$ 和 $\bar{f} : E \to E'$ 是流形间的光滑映照, 且满足如下交换图*

$$
\begin{tikzcd}
E \arrow[r, "\bar{f}"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\
M \arrow[r, "f"] & M'
\end{tikzcd}
$$

辅以要求 $\bar{f}$ 是 G-等变的.

一个最简单的例子是所谓的平凡主丛, 即 $M' = \{\mathrm{pt}\}$, 而 $E' = G$, 此时 $E = M\times G$. 我们称一个主丛是平凡的, 若其同构于平凡主丛. 一个关键性的结论之为主丛若有整体截面则必平凡, 这是其与向量丛十分不同之处, 也是尤其因为其纤维为 Lie 群而非向量空间.(类似的结论应当是若秩为 k 的向量丛有 k 个处处线性无关的整体截面, 则向量丛为平凡丛.)

这样的交换图使人自然想到拉回的构造, 当然, 主丛上同样可以构造拉回, 并且有所有拉回都有的泛性质.

定义 1.3. *给定主丛 (E, π, M), 光滑流形 M' 和光滑映照 $f : M' \to M$, 我们定义拉回 $f^*(E) = \{(p, x) \in M'\times E | f(p) = \pi(x)\}$, 从而 $\bar{f}$ 和 π' 都为自然的投影.*

泛性质则以如下交换图表示, 于不言中道尽一切

$$
\begin{tikzcd}
\tilde{E} \arrow[dr, dashed, "\exists!\phi"] \arrow[drr, bend left, "\tilde{f}"] \arrow[ddr, bend right, "\tilde{\pi}"'] & & \\
& f^*(E) \arrow[r, "\bar{f}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
& M' \arrow[r, "f"] & M
\end{tikzcd}
$$

对于主丛这样的结构, 同样可以和其他我们在几何中熟悉的构造相结合, 实际上, 这便是有着极为简单的对应, 马上我们来给出其上同调描述. 由于我们假定大家了解 Čech 上同调的定义, 否则可以参见 [4], 此番主要的结果将被写为如下之定理:

定理 1.1. *M 上的 G-主丛一一对应于 $\check{\mathrm{H}}^1(M,G)$ 中的元素.*

取开覆盖 $\mathcal{U}=(U_i)_{i\in I}$ 平凡化 E, 也即是存在 G-等变微分同胚 $\varphi_i:\pi^{-1}(U_i)\to U_i\times G$. 这样在两个开集相交的地方 $U_i\cap U_j$ 上就有

$$\varphi_j\circ\varphi_i^{-1}:(U_i\cap U_j)\times G\to(U_i\cap U_j)\times G,$$

在元素上可以具体记为 $(x,g)\mapsto(x,c_{ij}(x)g)$. 我们作出如下之交换图, 以便宜理解此构造:

$$\begin{tikzcd}
(U_i\cap U_j)\times G \arrow[rr, "(\mathrm{id},c_{ij})=\varphi_j\circ\varphi_i^{-1}"] & & (U_i\cap U_j)\times G \\
& \pi^{-1}(U_i\cap U_j) \arrow[ul, "\varphi_i"] \arrow[ur, "\varphi_j"'] &
\end{tikzcd}$$

我们的精神是: 这组资料 (U_i,c_{ij}) 在一定条件下就能够决定主丛的结构, 因为本质上它蕴含了如何将平凡的局部粘合成整体的信息. 易于验证上面定义的 (U_i,c_{ij}) 就给出了一个 $\check{\mathrm{H}}^1(M,G;\mathcal{U})$ 中的上同调类. 而更细的覆盖 $\mathcal{U}'$ 当然也平凡化此主丛, 因此也对应 $\check{\mathrm{H}}^1(M,G;\mathcal{U}')$ 中的元素, 这样便给出 $\check{\mathrm{H}}^1(M,G)$ 中的一个上同调类.[1]

反过来, 一个 $\check{\mathrm{H}}^1(M,G)$ 中的元素也给出一个 M 上的 G-主丛. 构造的想法是先取出每一片覆盖 U_i 上对应的 $U_i\times G$, 再用上同调中的元素 c_{ij} 将其粘合成所要的 E. 具体地说, 先直接构造无交并 $E'=\coprod_i(U_i\times G)$, 然后我们的工作便是商掉不同开集之间的转换关系. 今在其上定义等价关系: 设 $(x,g_i)\in U_i\times G$, 且设 $(x,g_j)\in U_j\times G$, 即是 M 上同一个点但是分属于不同的开集上, 那么二者等价 $(x,g_i)\sim(x,g_j)$, 若存在 c_{ij}, 使得 $g_j=g_ic_{ij}(x)$. 由此可见余链 (cocycle) 条件 $c_{ij}\circ c_{ki}=c_{kj}$ 的意义便在于使得这个关系确实成为一个等价关系, 具体来说便是使得该关系具有传递性. 从而商空间 $E=E'/\sim$ 辅以其到第一个分量的投影便是一个 M 上的 G-主丛.

严格的论述只欠验证上面的两个构造是互逆的, 这边便留给有闲暇的读者完成.

2 联络与曲率

后面要用到向量值的微分形式, 为了保证在使用的时候心中没有磕绊感, 下面快速地列出一些必要的定义, 大部分东西与平时的微分形式无二致, 一些需要注意的状况会被详细解释.

定义 2.1. *设 V 是向量空间,M 是光滑流形, 其上的 V-值 k 次微分形式 ω 为 $\bigoplus_k TM\to V$ 的交错 k-线性的函数, 且在 $C^\infty(M)$ 意义下, 我们记这样的 ω 的集合是 $\Omega^k(M,V)$.*

这样的定义不过是为了与取基那样人为性的操作划清界限罢了, 实际上心中自知不过 $\omega=\omega_1e_1+\ldots+\omega_ke_k$, 其中 $(e_i)_{i=1,\ldots,k}$ 是为 V 的一组基, 而诸 $(\omega_i)_{i=1,\ldots,k}$ 为平常的微分形式. 从此, 算子 d, 各种诱导映照的定义全都是自然的. 我们在此关心一下楔积, 先给出定义

定义 2.2. *令 $\omega_1\in\Omega^k(M,V)$, 而 $\omega_2\in\Omega^l(M,W)$, 则 $\omega_1\wedge\omega_2\in\Omega^{k+l}(M,V\otimes W)$, 具体写为*

$$(\omega_1\wedge\omega_2)(X_1,\ldots,X_{k+l})=\sum_{\sigma\in\mathfrak{S}_{k+l}}\mathrm{sgn}(\sigma)\omega_1(X_{\sigma(1)},\ldots,X_{\sigma(k)})\otimes\omega_2(X_{\sigma(k+1)},\ldots,X_{\sigma(k+l)}),$$

其中 $\mathfrak{S}_p$ 表示 p 阶置换群, 群同态 $\mathrm{sgn}:\mathfrak{S}_p\to\{\pm1\}$ 是自然的符号映射.

注意到这和我们平时使用的楔积也是一致的. 在平时 $V=W=\mathbb{R}$, 则有自然的乘法 $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, 这将诱导 $\Omega^k(M,\mathbb{R}\otimes\mathbb{R})\to\Omega^k(M,\mathbb{R})$, 从而与寻常的状况符合. 以上定义出来的诸多概念的性质也与平时所用一样, 比如 $d^2=0$, 不需要担心.

[1] 实际上我们可以取 $\mathcal{U}$ 为所谓的好覆盖 (good cover), 即任意有限多个的交都可缩, 这样可以有 $\check{\mathrm{H}}^1(M,G;\mathcal{U})=\check{\mathrm{H}}^1(M,G)$.

对于 Lie 群, 我们需要回念到, 有个概念称为伴随表示, 实际上根本就是

$$\mathrm{Ad}: G \to GL(\mathfrak{g}), g \mapsto d(x \mapsto gxg^{-1}),$$

此处已经典范地将 $\mathfrak{g}$ 视为左不变向量场集合和视为单位元处切空间两种观点等同, 并一般都以后者为主.

有了充足的准备, 我们来考虑如何定义主丛上的联络! 在微分几何中, 联络的定义方式很多, 观点也很多, 比如为了构造类似于欧氏空间中的方向导数, 比如为了实现将向量进行平行移动. 具体到主丛上来说, 我们也有类似的想法, 本质上是为了得到光滑的移动. 考虑 $x \in E$, 则我们由于局部平凡化的存在, 可以确信 T_xE 应当由 M 的切空间和 G 的切空间组成. 极巧之处在于 G 恰为 Lie 群, 其不同点处的切空间通过群乘法的切映射架起同构, 总可以视之为 $\mathfrak{g}$. 但是 E 上没有乘法, 取而代之的是右作用, 因而这启发我们考虑 $v_x : \mathfrak{g} \to T_xE$ 为 $g \mapsto xg$ 的切映射, 因而我们可以得到此般的短正合列

$$0 \longrightarrow \mathfrak{g} \xrightarrow{v_x} T_xE \xrightarrow{\pi_*} T_{\pi(x)}M \longrightarrow 0$$

作为线性空间的短正合列, 分裂再自然不过了. 但是, 此般分裂必然在先验上只能逐点, 而几何上总需要光滑性, 因而我们的联络的存在是有必要的, 以特别的姿态出现. 对上述短正合列我们仍然需要定义一些东西, 称 $v_x(\mathfrak{g})$ 是垂直子空间, 而其一个补子空间 H_x 被称为水平子空间, 对应于 M 的切空间.

- 我们希望这些子空间能随着 x 的变化而光滑移动, 换言之, 我们实际上可以选取一个光滑函数 ω, 逐点上是为线性映射 $\omega(x) : T_xE \to \mathfrak{g}$, 使得这成为上述短正合列中的一个截面, 而 H_x 自然变成了核空间, 换言之, 我们选取了一个 $\omega \in \Omega^1(E, \mathfrak{g})$.
- 群作用在主丛的构造中不可忽视, 此处我们注意到我们选取的水平子空间可以被 G 中元素右作用从而移动, 我们应该希望满足某种要求, 具体而言, 定义右乘作用 $R_g : E \to E, x \to xg$, 则要求 $(R_g)_*H_x = H_{xg}$.

我们已然列完了心里觉得作为一个主丛的联络应当有的条件, 整理如下:

定义 2.3. *一个 G-主丛 (E, π, M) 上的联络 ω 是 $\Omega^1(E, \mathfrak{g})$ 中的元素, 并满足如下两条要求:*

- *任取 $x \in E$, 应当有 $\omega_x \circ v_x = \mathrm{id}$;*
- *令 $H_x = \mathrm{Ker}\omega_x$, 则 $\forall x \in E$, 要求 $(R_g)_*H_x = H_{xg}$.*

但是第二条要求中的定义并不足够代数, 不方便计算, 许多时候大家会用另一个 (在第一条成立时) 等价的表述:$R_g^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$. 下面通过简单的计算证明一下这两条的等价性, 并不困难.

证明. $\Rightarrow$: 注意到第二式对水平向量已然为 0, 故而实际上只需考虑 $v_x(X)$, 其中 $X \in \mathfrak{g}$. 考虑映射 $a \mapsto xag = xg \cdot (g^{-1}ag)$, 此之诱导了 $(R_g)_* \circ v_x = v_{xg} \circ \mathrm{Ad}(g^{-1})$, 准此即有 $R_g^*(\omega)(v_x(X)) = \omega_{xg}(v_{xg} \circ \mathrm{Ad}(g^{-1})(X)) = \mathrm{Ad}(g^{-1})(X)$.

$\Leftarrow$: 由于维数一致, 只需证 $(R_g)_*H_x \subset H_{xg}$, 任取一个 $X \in H_x$, 易见 $\omega_{xg}((R_g)_*(X)) = (R_g^*\omega)(X) = \mathrm{Ad}(g^{-1})(\omega(X)) = 0$, 从而 $(R_g)_*(X) \in \mathrm{Ker}\omega_{xg} = H_{xg}$. □

至于联络的存在性, 也可以和微分几何那样类似操作, 只需要使用单位分解与全体联络之集合凸这样的结论即可完工. 下面再给出几个相关的定义, 然后我们开始考虑定义曲率.

定义 2.4. *一个 $\omega \in \Omega^k(E, V)$ 称为水平的, 若只要存在 $i = 1, ..., k$, 有 X_i 是垂直的, 则 $\omega(X_1, ..., X_k) = 0$.*

定义 2.5. *设 ρ 是 Lie 群 G 的一个 V-表示, 称 ω 是 ρ-等变的, 若 $R_g^*\omega = \rho(g^{-1}) \circ \omega$. 特别若有 ρ 平凡, 则 ω 称为不变的. 我们把 ρ-等变的 k-形式集合记为 $\Omega^k(M, E_V)$.*

那么对于定义曲率, 这一问题有一些先验的经验, 那就是必须由联络直接决定, 并且是 2-形式. 在常见的向量丛微分几何中, 曲率直接被定义为 $d\omega+\omega\wedge\omega$. 但在这里碰到了十分微妙的问题, 即 $d\omega\in\Omega^2(E,\mathfrak{g})$, 而 $\omega\wedge\omega\in\Omega^2(E,\mathfrak{g}\otimes\mathfrak{g})$, 即两个微分形式取值的线性空间不同. 那么有什么典范的办法让 $\mathfrak{g}\otimes\mathfrak{g}$ 中的元素变成 $\mathfrak{g}$ 中的元素呢? 当然有的, 这便是 $\mathfrak{g}$ 作为 Lie 代数自带的 Lie 括号 $[\cdot,\cdot]$! 通过 Lie 括号, 我们把 $\omega\wedge\omega$ 的像记为 $[\omega,\omega]$. 注意到这里有一个两倍的问题, 我们给出此般定义

定义 2.6. *曲率形式* $F_\omega=d\omega+\frac{1}{2}[\omega,\omega]$.

对于曲率, 我们可以有如下两个定理表明其性质:

定理 2.1. *任何一个曲率形式* F_ω *为* $\Omega^2(M,E_{\mathfrak{g}})$ *中的水平的元素, 其中* $G\to GL(\mathfrak{g})$ *为自然的伴随表示.*

定理 2.2. *Bianchi 恒等式为* $dF_\omega=[F_\omega,\omega]$.

审视曲率的观点十分丰富, 这一视角实际上将曲率置于某种可积性的视角中. 我们知道, 在向量丛的微分几何中, 平坦性实际上对应于水平子空间的可积性, 而此处同样强调了水平与垂直子空间, 也同样有类似的结论. 而我们在一般的 Riemann 几何课程中也许联络的定义就是从协变导数出发, 此时联络就有不止一个直接被放在台面上关心的率, 除了曲率还有挠率, 比如 Levi-Civita 联络会要求挠率消失, 而这些东西都可以反映在古典微分几何中, 以真正的几何直观的姿态存在于肉眼可见的三维几何中. 然而在历史发展中, 前一种观点的存在也愈发不可无视, 例如在复几何中会有 Newlander-Nirenberg 定理说明近复结构的可积性对应一种称为 Nijenhuis 挠率的东西消失.

3 例子与其他相关

考虑平凡丛 $E=M\times G$, 并且还考虑一个 Lie 群上的 Maurer-Cartan 形式 $\omega_0\in\Omega^1(G,\mathfrak{g})$, 我们可以取为 $(\omega_0)_g=(L_{g^{-1}})_*$. 这一方面可以直接参考 [1, 第五章]. 实际上, 这已经可以视为 $G\to\{\mathrm{pt}\}$ 上的联络. 我们定义 $\pi_2: M\times G\to G$, 从而把 ω_0 转移到了 E 上, 是为 Maurer-Cartan 联络 $\omega_{\mathrm{MC}}=\pi_2^*\omega_0$.

我们来验证这实为一个联络, 通过对定义验证即可, 其中定义的第二条可以使用更为方便的等价命题, 如下:

- 任取 $x=(p,g)$, 则此时 $v_x=d(a\mapsto(p,ga))$ 恰好就是左乘一个 g 对应的微分映射, 再经过 ω_{MC} 为左乘一个 g^{-1} 的微分映射, 则确实复合后为恒同;
- 注意到 π_2^* 和 R_g^* 和 $\mathrm{Ad}(g^{-1})$ 的交换性, 实际上只需要验证 ω_0 满足定义第二条的等价命题. 此时只需要简单的计算即有

$$\begin{aligned}R_g^*(\omega_0)_a&=(\omega_0)_{ag}\circ(R_g)_*=(L_{(ag)^{-1}})_*\circ(R_g)_*=(L_{g^{-1}})_*\circ(L_{a^{-1}})_*\circ(R_g)_*\\&=\mathrm{Ad}(g^{-1})\circ(L_{a^{-1}})_*=\mathrm{Ad}(g^{-1})\circ(\omega_0)_a.\end{aligned}$$

接下来计算一下对应的曲率, 只需注意到 $F_{\omega_{\mathrm{MC}}}=\pi_2^*F_{\omega_0}$, 其中已经把 ω_0 视为 $G\to\{\mathrm{pt}\}$ 上的 Maurer-Cartan 联络了. 由**定理 2.2.**,F_{ω_0} 必然是水平的, 但是 G 上所有切向量都是垂直的, 因为底流形根本就是一个单点, 所以迫使其曲率为 0, 从而平凡丛上 Maurer-Cartan 联络是平坦的.

前面提到了主丛可以通过表示构造伴丛, 同样, 主丛上的联络也可以通过表示推到伴丛上, 成为伴丛上的联络. 下面来简单介绍一下这一流程, 首先我们有一表示 $\rho: G\to GL(V)$, 这将诱导切映射 $d\rho:\mathfrak{g}\to\mathfrak{gl}(V)\simeq M_n(\mathbb{R})$, 此处 $n=\dim V$.

我们先前构造的 E_V 是一个正经的向量丛, 局部上同构于 $U\times V$, 其中 $U\subset M$ 是一个开集. 又回忆到 $E_V=(E\times V)/G$, 因而我们若取一组 V 的基 $(e_1,\dots,e_n)$, 且取一个局部截面 (规范选取)$s:U\to E$, 则这就得到了一个局部标架场 $([(s,e_1)],\dots,[(s,e_n)])$, 其中 $[\cdot]$ 表等价类.

通过局部规范选取, 我们有 $s^*\omega\in\Omega^1(U,\mathfrak{g})$, 进而 $d\rho\circ s^*\omega\in\Omega^1(U,\mathfrak{gl}(V))\simeq\Omega^1(U,M_n(\mathbb{R}))$, 如斯所得便是局部联络方阵. 这样一大通操作下来, 把主丛的联络变成了伴丛上的联络, 进而我们常规的微分几何理论就能介入了.

4 代数视角下的主丛

在 Grothendieck 拓扑被提出之后, 种种关于实和复流形的理论得以被迁移到代数的世界. 在此节中, 我们将简单介绍代数情况下的主丛, 即旋子 (torsor)[2]. 我们约定代数簇是域上分离有限型的概形, 但不要求它是整概形.

例 4.1. *考虑有限 Galois 扩张 K/k, $G=\mathrm{Gal}(K/k)$. 从而通过 G 在 K 上的作用,G 也作用在 $\mathrm{Spec}\,K$ 上. 虽然 $\mathrm{Spec}\,K$ 的底空间是一个点, 但作为概形我们还应该考虑这个点上的 "函数环"K. 可以验证 $K/G=k$(作为 k-线性空间), 从而我们可以将 $\mathrm{Spec}\,K\to\mathrm{Spec}\,k$ 视为 G-主丛.*

为了把主丛的定义迁过来, 我们还需要获得 Lie 群的类比, 即具有群结构的代数簇.

定义 4.1. *所谓代数群是一个代数簇 G/k, 满足对于任意域 K/k, 使得 G 的 K-有理点 $G(K)=\mathrm{Hom}(\mathrm{Spec}\,K,G)$ 有群结构. 对一个概形 S, 其上的群概形是指概形 G, 满足对于任意 S 上的概形 T, 有 T-点 $G(T)=\mathrm{Hom}(T,G)$ 有群结构.*

如果读者愿意接受更多范畴语言, 这个定义就是在说 G 是 S 上群概形, 当且仅当, 在 Yoneda 嵌入 $\mathrm{Sch}/S\hookrightarrow\mathrm{Sh}/S$ 下 G 对应群层.

例 4.2. *在前面考虑的实情况下, 一个点上的 G-主丛 $G\to\{\mathrm{pt}\}$ 显然只能是平凡的, 也就是 G 通过右乘作用在自身上. 这一点也可以由 Čech 上同调 $\check{\mathrm{H}}^1(\{\mathrm{pt}\},G)=0$ 看出. 然而, 在代数世界里因为可以改变基域, 我们将会有很丰富的主丛 $G\to\mathrm{Spec}\,k$. 虽然还没有给出具体定义, 但是和之前的论证完全一样, 主丛会和系数在 G 中的一阶上同调对应, 而 $\check{\mathrm{H}}^1_{et}(k,G)=\mathrm{H}^1_{et}(\mathrm{Spec}\,k,G)$ 一般不会是 0.(可见 [9].)*

但是注意群上同调 $\mathrm{H}^1(k^s,G)=0$, 其中 k^s 是可分闭包. 换言之, 如果把 G 拉回到 k^s 上得到 $G_{k^s}\to\mathrm{Spec}\,k^s$, 则它必将是平凡主丛. 这提示我们 $G\to k$ 的不平凡性实则来源于基域的绝对 Galois 群 $\mathrm{Gal}(k^s/k)$ 中的元素带来的扭转 (twist).

例 4.3. *如前所述, 考虑有限 Galois 扩张 K/k, 有 Galois 群 $G=\mathrm{Gal}(K/k)$ 和扩张次数 $n=[K:k]$. 根据可分扩张的本原元定理, 总可不妨设 $K=k(\alpha)=k[T]/(p(t))$, $p(t)$ 是 α 的极小多项式, 在可分闭包上分裂为 $p(T)=\prod_i(T-\alpha_i)$. 这样拉回到 k^s 上我们有*

$$K\otimes_k k^s=k^s[T]/(p(t))=k^s[T]/\prod_i(T-\alpha_i)=\prod_i k^s[T]/(T-\alpha_i)=(k^s)^n.$$

对于 $\sigma\in G$, 设 K 在 k 上的一组基为 $\{\beta_i\}$, 它在上述同构下对应 $(k^s)^n$ 的一组基 $\{e_i\}$, G 在 $\{\beta_i\}$ 上传递地作用. 如果 $\sigma\beta_i=\beta_j$, 那么 σ 在 $(k^s)^n$ 上的作用由 $\sigma x_i=x_j$ 给出. 这便是 k^s 上的常群概形 G 在自己上的左乘作用, 也就是说, 主丛 $\mathrm{Spec}\,K\times_{\mathrm{Spec}\,k}\mathrm{Spec}\,k^s$ 是 $\mathrm{Spec}\,k^s$ 上平凡 G-主丛.

例 4.4. *考虑域 k 上的椭圆曲线 E, 我们知道它具有群结构. 对于任意与 $\mathrm{char}\,k$ 互素的正整数 n,$[n]:E\to E$ 是自乘 n 次映射 $x\mapsto nx$. 态射 $[n]$ 的核, 即是 E 的 n-挠点记为 $E[n]$, 这*

[2]这个名词似乎还没有统一的中文翻译.

是一个 0 维代数群. 可以证明 $[n]:E\to E$ 是一个 $E[n]$-主丛.

一般地, 将上面的 E 更替为 *Abel* 簇 A, 这些结论同样成立.*Abel* 簇是指整的紧合 *(proper)* 代数群, 可以证明它一定是光滑的, 并且群结构是交换的.

前面的一些例子为我们积累了一定的感觉, 于是如今我们来考虑代数情况下主丛的定义 [3]. 一个重要的问题是代数簇上的 Zariski 拓扑不够精细, 难以很好地类比欧氏拓扑: 例如前面的**例 4.1.** 中底空间 $\operatorname{Spec} k$ 在 Zarski 拓扑下为一个点, 直接仿照前面定义则其上当然只能有平凡的主丛. 为了解决这一问题,Grothendick 建立了所谓位形 (site) 的概念. 他的观点是对于 (概形, 代数簇, 流形等)X, 研究许多问题 (如其上的层和主丛等) 只要开集之间的包含与交的信息就够了, 而这组信息很容易推广到更普适的情形. 具体来说, 对于某个 (对象为概形的) 范畴 C 的对象 X, 一族态射 $\{X_i\to X\}$ 如果满足其像的并为 X, 则称为 X 的覆盖. 此范畴 C 上的一个位形 [4](site) 是指对 C 中的任何一个对象 X, 都有一族 X 的覆盖 C_X, 满足以下三点:

1. 如果 $U\to X$ 是同构, 则 $\{U\to X\}\in C_X$.
2. 如果 $\{U_i\to X\}_i\in C_X$, 而 $V\to X$ 是一个 C 中态射, 那么 $\{U_i\times_X V\to V\}_i\in C_V$.
3. 如果 $\{U_{ij}\to V_i\}_j\in C_{V_i}$, 且 $\{V_i\to X\}_i\in C_X$, 则 $\{U_{ij}\to X\}_{i,j}\in C_X$.

例 4.5. 设 S 是概形, (Sch/S) 是 S 上概形构成的范畴. 此处给出一些位形的可能之例子:

- $(\mathrm{Sch}/S)_{et}$ 的对象为全部 S 上概形, 态射为平展态射 [5].
- $(\mathrm{Sch}/S)_{fppf}$[6] 的对象为全部 S 上概形, 态射为忠实平坦有限展示态射.
- X_{et} 的对象为全部 X 上概形 Y, 满足结构态射 $Y\to X$ 是平展的, 态射自然可定义.

此三范畴都可以自然地定义每个对象的覆盖, 从而成为位形, 分别称为大平展位形, *fppf* 位形, 小平展位形.

关于位形的更多细节请读者参阅 [10]. 下面即刻来给出旋子的定义.

定义 4.2. 设 X,Y 是 S 上的概形, 且设 G 是一个 X 上的群概形. 如果态射 $f:Y\to X$ 满足存在一个 X 的 *fppf* 覆盖 $\{U_i\to X\}$, 满足对任意 i 有 G-等变的同构 $U_i\times_X Y\simeq U_i\times G$, 则称 $f:Y\to X$ 为一个 G-旋子 *(torsor)*.

注记. 为了简洁, 我们考虑 $U=\coprod U_i$, U 也是 X 的 *fppf* 覆盖. 这样, 上面的定义就等价于如果存在一个 X 的 *fppf* 覆盖 $U\to X$, 有 G 等变的同构 $U\times_X Y\simeq U\times_S G$, 则称 $f:Y\to X$ 为一个 G-旋子.

例 4.6. 对于拓扑群 G, 设流形 M 上 G-主丛的集合为 $\mathcal{P}rinc(M)$. 我们可以有 G-主丛的分类空间 BG, 满足对任意流形 M, 映射的同伦类 $[M,BG]\cong\mathcal{P}rinc(M)$, 即用同伦类分类了主丛. 同样对于 S 上群概形 G 和 S 上概形 X, 记 X 上的 G-旋子集合为 $F(X)$, 存在分类空间 $BG=[S/G]$ 满足 $\mathrm{Hom}(X,BG)\cong F(X)$, 不过这里的 BG 通常不再是概形, 而是代数叠 *(algebraic stack)*, 详见 *[7, 8.1]*. 准确地说, 分类空间 BG 表出了函子 F, 或者 BG 是 G-旋子的细模空间 *(fine moduli space)*.

下面给出旋子在有理点问题的一些应用, 并且导出弱 Mordell-Weil 定理.

定义 4.3. 设 X 是域 k 上的代数簇,K 是 k 的扩域, 则称态射 $\operatorname{Spec} K\to X$ 为 X 的一个 K-有理点. 代数簇 X 的全部 K-有理点记为 $X(K)$. 而数域 *(number field)* 是指 $\mathbb{Q}$ 的有限扩张.

[3]若读者对概形和范畴语言不熟悉, 可跳过此段直接阅读旋子的定义.

[4]这个名词也尚没有统一翻译.

[5]平展态射是指平坦非分歧态射, 它在局部上模拟了拓扑中的有限覆叠空间, 见 [6].

[6]缩写 fppf 来自法语 fidèlement plate de présentation finie, 即忠实平坦有限展示.

集合 $X(K)$ 的意义就是定义 X 的方程组在域 K 中的全部公共解.

命题 4.1. *设 $f: T \to k$ 是一个 G-旋子, 那么它平凡当且仅当 T 有一个 k-有理点.*

证明. 注意到 k 上的旋子的 k-有理点是该旋子的一个截面, 因此必为平凡旋子. □

设 X, Y 是 k 上代数簇, 设 G 是 k 上代数群, 映照 $f: Y \to X$ 是一个 $G_X = G \times_k X$-旋子. 假设 x 是 X 的 k-有理点, 那么通过 $x: \operatorname{Spec} k \to X$ 拉回 Y, 我们得到 $Y_x \to x = \operatorname{Spec} k$. 这是一个 k 上的 G-旋子, 因此对应了一个 $\mathrm{H}^1(k, G)$ 中的元素. 这就是说, 我们可以通过 $\mathrm{H}^1(k, G)$ 中的元素来分类 $X(k)$, 即

$$X(k) = \bigcup_{\sigma \in \mathrm{H}^1(k,G)} \{x \in X(k) : \Phi(Y_x \to k) = \sigma\},$$

这里 $\Phi : \{k \text{ 上 } G\text{– 旋子 }\} \to \mathrm{H}^1(k, G)$ 是前面所述旋子和一阶上同调的一一对应.

命题 4.2. *我们有集合间的相等 $\{x \in X(k) : \Phi(Y_x \to k) = \sigma\} = f_\sigma(Y_\sigma(k))$, 其中 $f_\sigma : Y_\sigma \to X$ 是 σ 给出的扭转 (twist).*

证明并不困难, 但是用到 twist 具体的性质, 可见 [8, 8.4.].

从而有

$$X(k) = \bigcup_{\sigma \in \mathrm{H}^1(k,G)} f_\sigma(Y_\sigma(k)).$$

人们很早就知道了椭圆曲线是 Abel 簇, 那么自然会关心它的具体群结构.Mordell 在 1922 证明了对于数域 K 上的椭圆曲线 E, 有 $E(K)$ 是有限生成的, 而后 Weil 对 Abel 簇证明了同样的结论. 通常的证明分为两部分, 第一部分就是下面的

定理 4.3 (弱 Mordell-Weil 定理)**.** *设 K 是数域,A 是 K 上的 Abel 簇, 则对任意正整数 n, 有 $A(K)/nA(K)$ 是有限群.*

证明. 将 A 看作 $(\operatorname{Spec} K)_{\mathrm{et}}$ 上的层. 对 $\operatorname{Spec} K$ 上的平展拓扑, 有 n 倍态射 $[n] : A \to A$ 是满射, 因此有正合列

$$0 \to A[n] \to A \xrightarrow{n\cdot} A \to 0,$$

从而诱导上同调群的长正合列

$$\cdots \to A(K) \xrightarrow{n\cdot} A(K) \xrightarrow{\partial} \mathrm{H}^1(k, A[n]) \to \cdots.$$

于是 $A(K)/nA(K) = \operatorname{Coker}(A(K) \xrightarrow{n\cdot} A(K)) = \operatorname{Im}(A(K) \xrightarrow{\partial} \mathrm{H}^1(K, A[n]))$, 只要证明 ∂ 的像是有限的.

然而根据**例 4.4.**, 此 $A(K) \xrightarrow{n\cdot} A(K)$ 可以看作 $A[n]$-旋子, 故而有 $\partial' : A(K) \to \mathrm{H}^1(K, A[n])$. 可以验证 $\partial' = \partial$. 故而只要证 $\operatorname{Im}(\partial')$ 有限.

对 K 上的簇 X 和 G-旋子 $f: Y \to X$, 我们定义 Selmer 集为

$$\operatorname{Sel}_f(k, G) := \{\sigma \in \mathrm{H}^1(k, G) : \sigma_v \in \operatorname{Im}(X(k_v) \to \mathrm{H}^1(k_v, G), \forall v \in \Omega_k)\}.$$

我们有如下结论, 详见 [8, 8.4.6]:

命题 4.4. *如果 X 是紧合的, 则 $\operatorname{Sel}_f(k, G)$ 有限.*

并且再注意到有 $\{\sigma \in \mathrm{H}^1(k, G) : Y_\sigma(k) \neq 0\} \subset \{\sigma \in \mathrm{H}^1(k, G) : Y_\sigma(k_v) \neq 0, \forall v \in \Omega_k\} = \operatorname{Sel}_f(k, G)$, 其中 Ω_k 是 k 的全部素位的集合, 就完成了证明. □

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紧阿贝尔群上最小扩张集的稳定性定理的新证明

荆一凡

摘 要

最近, 利用调和分析工具, 陶哲轩给出了紧阿贝尔群上的拥有接近最小扩张的两个紧集合的结构刻画. 这个结果可以看作一个连续版本的 Freiman 式定理. 在本文中, 我们先简要介绍这个问题的历史背景与最近的一些进展, 并给出陶的定理的一个新证明.

1 群上的小扩张集合

在现代组合学中, 一个重要的研究问题就是研究拥有"小扩张"的集合的结构. 我们假设 G 是一个群, $s(\cdot)$ 是 G 中的某个度量集合大小的函数 (如集合的势、Haar 测度、Banach 密度, 等). 对于 G 中的集合 A, B, 我们定义

$$AB = \{ab \mid a \in A, b \in B\}.$$

当 G 是阿贝尔群时, 我们通常用 + 来表示 G 中的二元运算, 这时候我们用 $A + B$ 来表示集合 $\{a + b \mid a \in A, b \in B\}$. 对正整数 n, 我们定义 $A^n = \{a_1 \cdots a_n \mid a_i \in A\}$, 以及当 G 为阿贝尔群时, $nA = \{a_1 + \cdots + a_n \mid a_i \in A\}$. 一般的, 当 $s(A^2)/s(A)$ 比较小时, 我们称 A(在 $s(\cdot)$ 的意义下) 具有"小扩张". 我们先举一个特殊的例子.

当我们从整数环 $\mathbb{Z}$ 中随机的选取有限多个元素组成 A 时, 容易看到, 很大的概率下, $|A+A|$ 的量级为 $|A|^2$. 从另一个角度, 如果我们知道存在一个正实数 K 满足 $|A+A| \le K|A|$, 那么 A 一定满足一些特殊的结构. 著名的 Freiman 定理告诉我们, 如果 $|A+A| \le K|A|$, 那么 A 被一个维数和长度均被关于 K 的函数限制的广义算术级数 (generalized arithmetic progression) 中. 这里广义算术级数 (有时也称为高维等差数列) 是指

$$\left\{a + \sum_{i=1}^{D} k_i d_i \,\middle|\, k_i \in \mathbb{Z}, 0 \le k_i \le M\right\},$$

其中 D 代表维数, 集合的势为这个算术级数的长度.

对于一般的有限阿贝尔群 G, Green 和 Ruzsa 证明了类似的结果: 如果 A 是 G 的一个非空有限集, 且 $|A+A| \le K|A|$, 那么 A 被一个维数和长度有界的陪集级数 (coset progression) 包含; 这里一个陪集级数是形如 $H+P$ 的集合, 其中 H 是 G 的一个陪集, P 是一个广义算术级数.

算术级数, 以及更一般的陪集级数, 某种程度上可以看作是子群这个数学对象的推广. 因此我们可以稍微不严谨的去理解这个现象, 即有限阿贝尔群上的小扩张集合, 结构上都像一个子群. 这个现象在其他的代数结构上也一般成立. 例如我们考虑多项式环 $F[x, y]$, 其中 F 是某个特征零代数闭域. 对于非退化的多项式 $P \in F[x, y]$, 以及 F 上的有限集 A, B, 定义

$$P(A, B) = \{P(a, b) \mid a \in A, b \in B\}.$$

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稿件信息: 4 月 7 日来稿, 5 月 10 日接受. 稿件编辑: 包恺成.

交换环可以看作具有两个阿贝尔群的结构. 如果 $|P(A,A)| < K|A|$, 那么多项式 P 就一定要在这两个阿贝尔结构中做出 "选择": 在这种情况下, 一个对称的 Elekes–Ronyai 定理 [5] 告诉我们, P 要么是 $f(r(x)+cr(y))$, 要么是 $f(r(x)r(y)^n)$, 这里 $f, r \in F[x]$, $c \in F$, $n \in \mathbb{Z}$. 简单的说, P 要么被加法阿贝尔群控制, 要么被乘法阿贝尔群控制.

当 G 是非阿贝尔群时, 类似的现象仍然存在. 即如果 G 中存在小扩张集合 A, 那么 A 一定被一个近似阿贝尔的近似群结构控制. 这个现象可以从 Gromov 的关于多项式增长群的定理 [4] 看出, 完整的刻画来自 Breuillard–Green–Tao 的关于近似群 (approximate group) 的结构定理 [1]. 一般的非阿贝尔群中, 绝大部份的子集都不会有小扩张. 这种现象一般称作 "group growth", 目前在非交换概率, 群论, 数论, 组合等方向中有非常丰富的应用. 本文我们将集中在阿贝尔群结构上.

2 群上的最小扩张集合

当群为 $\mathbb{Z}$ 时, 可以很容易看到, 对任意两个非空有限集 A, B, 我们都有 $|A+B| \geq |A| + |B| - 1$; 这也给出了这种情况下集合的 "最小扩张". 对于一般的群, 得到扩张的最小值并不容易. 对于循环群 $\mathbb{Z}/p\mathbb{Z}$, 最小扩张由著名的 Cauchy–Davenport 定理给出, 即对于非空集合 A, B, 我们有 $|A+B| \geq \min\{|A|+|B|-1, p\}$.

对于一般的局部紧阿贝尔群 G, 假设 μ 是 G 上的一个 Haar 测度, A, B 是两个非空紧集合, Kneser 证明了

$$\mu(A+B) \geq \min\{\mu(A)+\mu(B)-\mu(H), \mu(G)\},$$

其中 H 是 G 中测度最大的真子集. Kneser 定理在局部紧幺模非阿贝尔群的推广由 Kemperman 给出.

拥有最小扩张或者接近最小扩张的集合一般拥有更强的结构性质. 例如 Freiman $3k-4$ 定理指出, 如果 $A \subseteq \mathbb{Z}$ 且 $|A+A| \leq 3|A|-4$, 那么 A 被一个长度至多 $|A+A|-|A|$ 的等差数列 (即维度为 1 的算术级数) 包含. 在 $\mathbb{Z}/p\mathbb{Z}$ 中, Vosper 定理指出, 若两个非空有限集 A, B 满足 $|A+B| = |A|+|B|-1$, 那么 A 与 B 是两个公差相等的等差数列 (这里需要去除平凡情况, 如 $|A| = 1$ 时).

我们这里考虑当 G 是一个连通的紧阿贝尔群的情形, μ_G 是群上的 normalized Haar 测度 (即群 G 的测度为 1). 由于 G 是连通的, 因此它不包含开子群, 于是对任意两个非空紧集 A, B, Kneser 定理给我们如下不等式

$$\mu(A+B) \geq \min\{\mu(A)+\mu(B), 1\}.$$

我们在本文中将上述不等式称作 Kneser 不等式, 它会在后文的证明中起到重要作用. 上述不等式中等号成立时 (即 A, B 拥有最小扩张), 集合的结构刻画由 Kneser 给出. 当等号几乎成立时, 即 $\mu_G(A+B) < \mu_G(A)+\mu_G(B)+\delta$ 时, 对充分小的 δ, 集合的结构刻画最近由陶哲轩给出. 在叙述这个结果之前, 我们先引入一个定义.

定义 2.1 (Bohr 集)**.** G 是一个局部紧阿贝尔群, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ 是一维环面, $\chi : G \to \mathbb{T}$ 是一个连续满同态. 假设 $I \subseteq \mathbb{T}$ 是一个紧区间. 我们称 $\chi^{-1}(I)$ 是 G 的一个 *Bohr* 集. G 上的两个 Bohr 集合 $\chi_1^{-1}(I)$ 和 $\chi_2^{-1}(J)$ 是平行 *Bohr* 集如果我们有 $\chi_1 = \chi_2$.

假设 I, J 是 $\mathbb{T}$ 上的两个紧区间且满足 $\mu_{\mathbb{T}}(I)+\mu_{\mathbb{T}}(J) < 1$. 容易看出, $\mu_{\mathbb{T}}(I+J) = \mu_{\mathbb{T}}(I) + \mu_{\mathbb{T}}(J)$. Kneser 关于最小扩张集合的定理 [7] 在说, 最小扩张集合本质上就来自于一维环面的紧区间.

定理 2.2 (Kneser)**.** *G 是一个紧阿贝尔群, A, B 为 G 上的两个非空紧集. 如果 $\mu_G(A+B) = \mu_G(A)+\mu_G(B)$, 则下面的情形中必有一个发生:*

1. $\min\{\mu_G(A), \mu_G(B)\} = 0$;
2. $\mu_G(A)+\mu_G(B) = 1$;
3. *A, B 是两个平行 Bohr 集.*

对于拥有接近最小扩张的集合 A, B, 其结构刻画就困难的多. 一般的, 稳定性定理不一定总是存在. 所谓稳定性定理, 在这里即表示拥有接近最小扩张的集合的结构, 应该接近拥有最小扩张的集合的结构. 这个定理最近由陶哲轩 [8] 给出.

定理 2.3 (Tao)**.** *对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得下述成立. G 是一个紧阿贝尔群, A, B 为 G 上的两个紧集, 满足 $\min\{\mu_G(A), \mu_G(B)\} > \varepsilon$ 且 $\mu_G(A)+\mu_G(B) < 1-\varepsilon$. 如果我们有*

$$\mu_G(A+B) < \mu_G(A)+\mu_G(B)+\delta,$$

那么存在一个连续满同态 $\chi : G \to \mathbb{T}$, 以及两个紧区间 $I, J \subseteq \mathbb{T}$, 满足

$$\mu_G(A \triangle \chi^{-1}(I)) < \varepsilon, \quad \mu_G(B \triangle \chi^{-1}(J)) < \varepsilon.$$

Tao 的原始证明应用了精巧的傅立叶分析以及非标准分析的技术, 有 30 页. 这里我们应用初等技术给出一个简单的新证明. 这个证明来自于最近作者和 Chieu-Minh Tran 关于非阿贝尔局部紧群上最小和接近最小扩张集的结构刻画的其中一步. 关于非阿贝尔群上的结果的背景介绍, 有兴趣的读者可以去读 [6].

3 证明的预备

证明的主要思路是使用归纳法对群 G 的 "维数" 进行归纳. 为了使维数这个概念有意义, 我们先将问题转化为李群的问题. 这一步我们要应用如下的紧群的结构定理. 局部紧群的结构刻画问题, 即希尔伯特第五问题, 完整的解答来自 Gleason[3] 和 Yamabe[9]. 这里我们只需要用连通紧阿贝尔群的版本, 来自 von Neumann 定理 [10] 的一个特殊情况.

定理 3.1 (von Neumann)**.** *假设 G 是一个连通紧阿贝尔群. 对任意包含单位元的 G 中的开集 U, 存在一个正规紧子群 $H \subseteq U$, 使得 $G/H \cong \mathbb{T}^d$.*

假设 H 从定理3.1得到的 G 的紧正规子群. 我们有如下的短正合列

$$1 \to H \to G \xrightarrow{\pi} \mathbb{T}^d \to 1.$$

通过选取连通分支, 我们不妨假设 H 也是连通的.

为了应用归纳法, 我们希望 A 与 B 在 $\mathbb{T}^d$ 上的投影也具有接近最小的扩张. 于是作为第一步, 通过选取 H, 我们希望 A 与 B 在 $\mathbb{T}^d$ 上的投影具有相对小的测度. 下面的引理满足了我们的需求. 引理的证明仅用到了定义以及紧性, 在这里略去.

引理 3.2. *假设 $\delta < \min\{\mu_G(A), \mu_G(B)\}$. 如果 $\max\{\mu_G(A), \mu_G(B)\} < 1/12$, 那么存在 G 的连通紧子群 H, 使得 $G/H \cong \mathbb{T}^d$, 且 $\max\{\mu_{G/H}(\pi A), \mu_{G/H}(\pi B)\} < 1/4$.*

注意到引理3.2中要求了 A, B 在 G 中的测度较小. 为了达到这个要求, 我们需要用到下述连续版本的 Dyson e-变换. 引理的证明仅用到了容斥原理.

引理 3.3. *假设 A, B_1, B_2 为 G 中的非空紧集, 满足*

$$\mu_G(A+B_1) < \mu_G(A)+\mu_G(B_1)+\delta_1, \quad \mu_G(A+B_2) < \mu_G(A)+\mu_G(B_2)+\delta_2,$$

且 $\mu_G(B_1\cap B_2)>0$, $\mu_G(B_1\cup B_2)<1-\mu_G(A)-\delta_1-\delta_2$. 那么我们有

$$\mu_G(A+(B_1\cap B_2))<\mu_G(A)+\mu_G(B_1\cap B_2)+\delta_1+\delta_2,$$

以及

$$\mu_G(A+(B_1\cup B_2))<\mu_G(A)+\mu_G(B_1\cup B_2)+\delta_1+\delta_2.$$

证明. 注意到对任意 $x\in G$ 我们都有

$$\mathbb{1}_{A+B_1}(x)+\mathbb{1}_{A+B_2}(x)\geq \mathbb{1}_{A+(B_1\cap B_2)}(x)+\mathbb{1}_{A+(B_1\cup B_2)}(x),$$

于是

$$\mu_G(A+B_1)+\mu_G(A+B_2)\geq \mu_G(A+(B_1\cap B_2))+\mu_G(A+(B_1\cup B_2)).$$

根据定理条件我们得到

$$2\mu_G(A)+\mu_G(B_1)+\mu_G(B_2)+\delta_1+\delta_2\geq \mu_G(A+(B_1\cap B_2))+\mu_G(A+(B_1\cup B_2)).$$

对不等式右边使用 Kneser 不等式, 命题得证. □

4 定理2.3的证明

在这节里, 我们假设有短正合列

$$1\to H\to G\xrightarrow{\pi}\mathbb{T}^d\to 1,$$

H 为连通的正规子群, A,B 为两个 G 上的集合, 满足 $\min\{\mu_G(A),\mu_G(B)\}>\varepsilon$, 且

$$\mu_G(A+B)<\mu_G(A)+\mu_G(B)+\delta,$$

同时 $\mu_{G/H}(\pi A)+\mu_{G/H}(\pi B)<1/2$. 这一步可以由引理3.2得到. 注意到为了应用这个引理, 我们要求 $\mu_G(A),\mu_G(B)<1/12$. 这个限制可以通过重复应用引理3.3来得到: 如果 $\mu_G(A+B)$ 小于 $\mu_G(A)+\mu_G(B)+\delta$, 那么将 A 替换为 $A'=A+g$, 上述不等关系对集合 (A',B) 仍然成立. 通过选取 g, 应用引理3.2, 我们可以让 $A\cap(A+g)$ 变得很小. 实际上, 通过应用 Fubini 定理我们可以看出, 我们总可以找到 g 使得 $\mu_G(A\cap(A+g))\leq\mu_G(A)^2$.

这节中, 我们的主要技术来自分部积分, 即算两次. 由于篇幅原因, 我们将在这里只给出一个证明的特殊情况, 即当集合 $A=B$ 时. 这个情况的证明已经包含了重要的新想法. 我们首先证明下面的关键引理:

引理 4.1. 假设 $2\mu_{G/H}(\pi A)<1$. 令 α 为 A 的最长的纤维长度, 即 $\sup_{a\in A}\mu_H(A\cap aH)$. 令 $\gamma=\max\{1,2\alpha\}$. 那么我们有

$$\mu_G(A+A)\geq\frac{4\alpha}{\gamma}\mu_{G/H}(\pi A_{(\alpha/\gamma,\alpha]})+4\mu_G(A_{(0,\alpha/\gamma]}).$$

在证明之前, 我们先明确一下证明中的记号: 我们总将 μ 用做 normalized Haar 测度, 其下标代表所在的群. 对于一个群 G 中的集合 A 以及一个 $[0,1]$ 区间的子集 I, 我们用 A_I 来表示 A 的所有纤维长度属于 I 的纤维集合. 注意到在本文中当我们提到纤维, 我们总是指集合与正规子群 H 的某个陪集的交. 对于 G 中的任意集合 X, 我们用 πX 表示 X 在 G/H 上的投影.

一般来说, 只要求 A 与 I 可测, 不一定会有 A 的纤维或者 A_I 可测. 这个问题可以通过一些标准的测度论技术来克服, 因此在本文的证明里我们假设我们关心的集合都是可测集.

我们下面给出引理4.1的简要证明:

证明. 对于任意 $x \in (0,1]$, 我们令 $C_x = (A+A) \cap \pi^{-1}(\pi A_{(x\alpha,\alpha]} + \pi A_{(x\alpha,\alpha]})$. 根据定义不难看出

$$\mu_G(A+A) \geq \mu_G(C_0).$$

另一方面,

$$\mu_G(C_0) = \mu_G(C_{1/\gamma}) - \int_0^{\frac{1}{\gamma}} \mathrm{d}\mu_G(C_x).$$

我们也有下面的不等式估计

$$\mu_G(C_{1/\gamma}) \geq \mu_{G/H}(\pi A_{(\alpha/\gamma,\alpha]} + \pi A_{(\alpha/\gamma,\alpha]}),$$

这是由于, 当 $\gamma = 1$ 时, 上述等式左右两边都为 0; 这里我们使用了 H 上的 Kneser 不等式).

同理, 通过考虑纤维的长度, 对于任意 $x, y \in \mathbb{R}^{>0}$ 满足 $x < y \leq 1/\gamma$, $\mu_G(C_x) - \mu_G(C_y)$ 一定至少是

$$2x\alpha\left(\mu_{G/H}(\pi A_{(x\alpha,\alpha]} + \pi A_{(x\alpha,\alpha]}) - \mu_{G/H}(\pi A_{(y\alpha,\alpha]} + \pi A_{(y\alpha,\alpha]})\right).$$

通过结合上述的两个不等式估计, 我们得到

$$\mu_G(C_0) \geq \mu_{G/H}(\pi A_{(\alpha/\gamma,\alpha]} + \pi A_{(\alpha/\gamma,\alpha]}) - \int_0^{\frac{1}{\gamma}} 2\alpha x \,\mathrm{d}\mu_{G/H}(\pi A_{(x\alpha,\alpha]} + \pi A_{(x\alpha,\alpha]}).$$

使用分部积分, 上述估计可以化简为

$$\mu_G(C_0) \geq 2\int_0^{\frac{1}{\gamma}} \mu_{G/H}(\pi A_{(x\alpha,\alpha]} + \pi A_{(x\alpha,\alpha]}) \,\mathrm{d}\alpha x.$$

根据假设 $2\mu_{G/H}(\pi A) < 1$, 我们在 G/H 上使用 Kneser 定理,

$$\mu_G(C_0) \geq 2\int_0^{\frac{1}{\gamma}} (\mu_{G/H}(\pi A_{(x\alpha,\alpha]}) + \mu_{G/H}(\pi A_{(x\alpha,\alpha]})) \,\mathrm{d}\alpha x.$$

再次使用分部积分, 我们有

$$\mu_G(C_0) \geq \frac{4\alpha}{\gamma}\mu_{G/H}(\pi A_{(\alpha/\gamma,\alpha]}) - 4\int_0^{\frac{1}{\gamma}} \alpha x \,\mathrm{d}\mu_{G/H}(\pi A_{(x\alpha,\alpha]}).$$

注意到 $\mathrm{d}\mu_{G/H}(\pi A_{(x\alpha,\alpha]}) = -\,\mathrm{d}\mu_{G/H}(\pi A_{(0,x\alpha]})$ 以及

$$\int_0^{1/\gamma} x\alpha \,\mathrm{d}\mu_{G/H}(\pi A_{(0,x\alpha]}) = \mu_G(A_{(0,\alpha/\gamma]})$$

引理得证. □

引理4.1告诉我们, 如果集合 A 的最长纤维长度小于 1/2, 那么 $\mu_G(A+A)$ 的大小至少是 $4\mu_G(A)$. 因此, 定理2.3的条件 $\mu_G(A+A) < 2\mu_G(A) + \delta$ 给力我们很强的结构刻画, 即 A 的最长纤维长度要接近 1; 实际上引理4.1可以给我们更多的结构刻画, 我们可以证明 A 中几乎所有的纤维的长度都要接近 1.

接下来为了书写方便, 我们用 $\mathfrak{d}_G(A)$ 来表示 $\mu_G(A+A) - 2\mu_G(A)$.

引理 4.2. *如果 $2\mu_{G/H}(\pi A) < 1$ 且 $\mathfrak{d}_G(A) < \mu_G(A)$. 那么存在一个可测集 $A' \subseteq G/H$ 使得 $\mathfrak{d}_{G/H}(A') < 2\mathfrak{d}_G(A)$ 并且 $\mu_G(A \triangle \pi^{-1}A') < \mathfrak{d}_G(A)/2$.*

证明. 引理4.1已经告诉我们, 如果 $\mathfrak{d}_G(A) < \mu_G(A)$ 那么 α, 即 A 中最长纤维的长度, 一定大于 1/2. 此时引理4.1变成了如下不等关系:

$$\mu_G(A+A) \geq 2\mu_{G/H}(\pi A_{(1/2,\alpha]}) + 4\mu_G(A_{(0,1/2]}).$$

我们取 $A' = \pi A_{(1/2,\alpha]}$. 注意到

$$\mathfrak{d}_G(A) \geq 2\mu_{G/H}(A') + 2\mu_G(A_{(0,1/2]}) - 2\mu_G(A_{(1/2,\alpha]}) = 2\mu_G(A \triangle \pi^{-1}A'),$$

这给出了 $\mu_G(A \triangle \pi^{-1}A') \leq \mathfrak{d}_G(A)/2$.

最后, 注意到 $\pi^{-1}(A'+A') = A_{(1/2,\alpha]} + A_{(1/2,\alpha]}$, 我们有

$$\mu_{G/H}(A'+A') \leq \mu_G(A+A) \leq 2\mu_G(A) + \mathfrak{d}_G(A) \leq 2\mu_{G/H}(A') + 2\mathfrak{d}_G(A),$$

引理证毕. □

引理4.2可以将问题几乎等价的转化到商群上去. 根据局部紧阿贝尔群的结构定理, 我们总可以将问题转化到环面 $\mathbb{T}^d$ 上; 而 $\mathbb{T}^d$ 上的定理2.3版本已经在 1998 年被证明 [2]. 通过使用这个在 $\mathbb{T}^d$ 上的结果, 我们这就完成了定理2.3的证明.

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Zariski 主定理和形式理论

包恺成

Abstract

在 Zariski 提出其“主定理”后, 其诸多不同推广也被冠以此称谓. 我们主要着眼定理 1.3, 1.4 中的两种形式. 第一部分先通过连通性引理 (引理 1.2) 证明 1.3, 1.4 以及 Stein 分解, 而后给出一些相关结论. 第二部分先通过形式函数定理证明引理 1.2, 随后转向更一般的形式概形理论, 介绍 Grothendieck 存在性定理, 并用之证明一些关于基本群特化 (specialization) 的结论.

Acknowledgement

这篇读书笔记原为作者的 “新生研讨课” 报告. 感谢许金兴教授对我的指导和建议, 让我对很多知识有更深的理解. 感谢曹阳教授解答我的一个问题. 感谢负责审稿的同学以及周潇翔学长耐心阅读了这篇笔记并提出中肯的意见, 指出许多不完善之处. 感谢陈恒宇学长帮助我绘制的精美插图. 感谢学校延迟了一年研讨课时限, 我从而得以学习更多知识并且将它们写进这篇笔记, 同时也感谢在学习过程中帮助和指导我的老师, 同学和网友们.

1 Zariski's Main Theorem

1.1 Introduction

We first recall some conventions. In this note a *variety* is an integral separated scheme of finite type over some field. Let X, Y be schemes, a *birational* map is a morphism f defined on a nonempty open set $U \subset X$ such that f induces an isomorphism from U to a nonempty open set $V \subset Y$. Two varieties over a field k are *birational* if and only if their function fields are isomorphic as extension fields of k.

Given a birational map $f : X \to Y$(where X, Y are varieties over field k), we call a point x *fundamental* if x lies outside the open set U where the birational map is defined. Let Γ_0 be the image of the graph morphism $U \to U \times Y$ and Γ be the closure of Γ_0 in $X \times Y$. For any closed set $Z \subset X$, its *total transform* under f is defined to be $p_2 \circ p_1^{-1}(Z)$ where p_1, p_2 are the projections of $\Gamma \subset X \times Y$ to X and Y, respectively. See [6, V.5] for details.

The original form of the theorem was stated and proved by Zariski, as the following

Theorem 1.1 (Zariski's main theorem, original form). *Let $f : X \to Y$ be a birational morphism between projective varieties and X is normal. Then if $x \in X$ is fundamental for f, then the total transform $f(x)$ is connected and of dimension at least 1.*

In [2], it was stated in a form concerning the connectedness of the fibers of a birational morphism by Grothendieck, as the consequence of the following lemma:

Lemma 1.2 (connectedness lemma). *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. If $f_*\mathcal{O}_X = \mathcal{O}_Y$, then the fibers of f are connected.*

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稿件信息: 本稿件于 2 月 7 日来稿, 4 月 11 日第一次修改, 5 月 19 日第二次修改, 5 月 27 日接受. 稿件编辑: 宋晨楷.

Theorem 1.3 (Zariski's main theorem, connectedness form)**.** *Let $f : X \to Y$ be a proper morphism of Noetherian integral schemes. If Y is normal and f is birational, then the fibers of f are connected.*

And using Lemma 1.2 we'll get another result namely the *Stein factorization* to factor a proper morphism into two simpler morphisms and study the property of fibers. By the factorization we reach a slightly different conclusion.

Theorem 1.4 (Zariski's main theorem)**.** *Let $f : X \to Y$ be a quasi-projective morphism of Noetherian schemes, and T be the points of X that are isolated in their fibers. Then T is open and $f|_T$ factors as $f = g \circ f'$, where f' is an open immersion and g is finite.*

Finally, in [3], the theorem was generalized to a stronger form about the quasi-finite morphisms. The proof is quite different from that of the previous versions, including hard commutative algebra.

Theorem 1.5 (Zariski's main theorem, final form)**.** *Let $f : X \to Y$ be a quasi-finite morphism of Noetherian schemes, then f factors as $f = g \circ f'$, where f' is an open immersion and g is finite.*

1.2 Elementary Examples

In this section we consider some simple examples for Theorem 1.3, and try to give a quite elementary proof in the case of projective curves. Notice that the morphisms of projective varieties(or schemes, in general) are all proper, hence the proper condition is automatically satisfied. In this section, all the varieties and schemes we considered are projective over some field k.

Example 1.6. One important class of birational morphisms is the blow up of varieties ([6, I.7, II.7]). Take the most naive situation, the blow up of a projective curve C at a point x. Let E be the strict preimage and

$$\pi : E \to C$$

is the canonical projection. We know that the fiber of π has more than 1 point only when x is singular and the map π fits the condition of Theorem 1.3 if and only if C is a non-singular curve, in which case π is just an isomorphism by the theory of blow up and has trivial fibers which are of course connected.

Now consider C with singular points, for example the nodal curve

$$Y^2Z = X^2(X + Z)$$

in $\mathbb{P}^2_k$. Since it only has one singular point $[0 : 0 : 1]$, we can just consider the affine piece $y^2 = x^2(x + 1)$ and its blowing-up at $O = (0, 0)$. E is isomorphic to a quadratic curve, with two points sent to O and 1-1 at other points. Write the point in the space $\mathbb{A}^2 \times \mathbb{P}^1$ by coordinate $(x, y); [u : v]$, we can write the equation of E explicitly. In the affine piece defined by $v \neq 0$, it is $(u/v)^2(x + 1) - 1 = 0$; in the piece $u \neq 0$, it is $(v/u)^2 - x - 1 = 0$.

One can check the following straightforward consequences:

- $E - \pi^{-1}(O)$ is open in E.
- $(\pi_*\mathcal{O}_E)_x$ is local ring, at any point $x \neq O$; or has 2 maximal ideals when $x = O$.

For the second claim, we compute directly by definition:

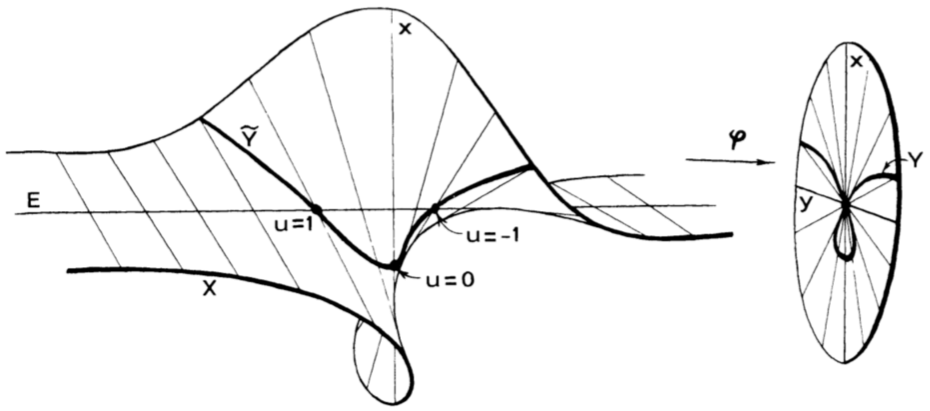
$$(\pi_*\mathcal{O}_E)_x = \varinjlim \pi_*\mathcal{O}_E(U) = \varinjlim \mathcal{O}_E(\pi^{-1}(U)) = \mathcal{O}_{E,\pi^{-1}(x)}$$

when $x \neq O$ or $O_{E,a} \oplus O_{E,b}$ where $\{a, b\}$ is the preimage of O.

These observations are special cases of the Propositions 1.11, 1.12, which claim that

- the points isolated in fibers form an open set
- the connected components of fiber $\pi^{-1}(y)$ correspond to the maximal ideals of $(\pi_* O_X)_y$.

□



Example 1.7. In the previous example, one can define a birational map $T : C \dashrightarrow E$ by π^{-1}, at everywhere of C expect O. The total transform of O is just the two points in $\pi^{-1}(O)$. C is not normal and $T(0)$ has 2 components. Readers can check this using the equation given above. □

Define a *curve* to be an integral proper variety of dimension 1 here. It is easy to show that a non-constant morphism between curves has fibers with finite points and we prove now that it is in fact a finite morphism, as a special case of the general result 1.17.

Proposition 1.8. *Let $f : X \to Y$ be a non-constant morphism of projective curves over field k, and X is non-singular, then f is finite.*

Proof. Take an affine open set $V = \operatorname{Spec} A \subset Y$, then A is an integral domain with $\operatorname{Frac}(A) = k(Y)$, the function field of Y. Let the preimage of V be $U \subset X$. Consider the integral closure B of A in $k(X)$. Since $k(X)$, $k(Y)$ are both of transcendence degree 1 over k, $k(X)/k(Y)$ is finite, hence B is finite over A as module. Let $U' = \operatorname{Spec} B$ be an affine variety of dimension 1, by valuative criterion([6, II.4]) we have a morphism $j' : U' \to X$ extending $\eta' = \operatorname{Spec} k(Y) \to X$. Notice that $U = f^{-1}(V)$ implies $U = X \times_Y V$, hence we have morphism $i : U' \to U$. Since $f' : U' \to V$ is finite, i is also finite([6, II.4]). Finite morphism is closed, so $i(U')$ is closed in U. The closed subset of U can only be finite points or U itself, but $j'(\eta') = \eta_X = \operatorname{Spec} k(X)$ so the only possible case is $i(U') = U$. By counting the size of fiber we have i is also injective, hence $U = U'$ which implies f is finite. □

$$\begin{tikzcd}
U' \arrow[dr, "i"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "j'"'] & & \\
& U \arrow[r, "f|_U"] \arrow[d, hook, "j"] & V \arrow[d, hook] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Proposition 1.9. *Let $f : X \to Y$ be a birational morphism of projective curves. If Y is normal, then f is a 1-1 map.*

Proof. Blow up the singularities of X we get $\pi : X' \to X$, which is also birational. Hence $f \circ \pi : X' \to Y$ is birational morphism of non-singular curves and hence an isomorphism. Our assertion follows. □

This is a baby version of 1.3.

1.3 Connectedness Theorem and Stein Factorization

In this section we use Lemma 1.2 to prove Theorem 1.3, 1.4 and some relevant results. The proof of Lemma 1.2 will be postponed, as a corollary of the formal function theorem.

Proof of Theorem 1.3. By Lemma 1.2, we just need to verify $f_*O_X = O_Y$. By the normal condition, $O_{Y,y}$ is integrally closed in its fraction field k, which is just the fraction field of $(f_*O_X)_y$, by the birational conditon. Therefore, $(f_*O_X)_y$ can only be $O_{Y,y}$ as module over it. So we get the desired equality. □

Remark 1.10. Using connectedness lemma one can also deduce a slightly stronger version ([7]):

Proposition 1.11. *Let $f : X \to Y$ be proper surjective morphism of integral Noetherian schemes, and Y is normal. Assume the generic fiber of f is geometrically connected, then f has geometrically connected fibers.*

When f birational, easy to see the generic fiber is geometrically connected. □

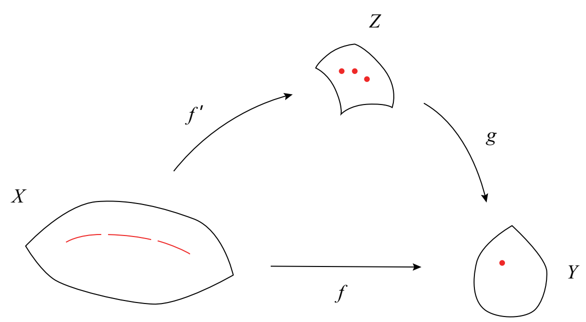
And by Lemma 1.2, we can derive the Stein factorization, using the relative spectrum which we denote by $\underline{\mathrm{Spec}}$. (One can find the construction of it on [6, Exercise II.5.17][18][17, 01LL])

Theorem 1.12 (Stein factorization)**.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes, then f factors as $f = g \circ f'$, where f' has connected fibers and g is finite.*

Proof. We have natural factorization of f :

$$X \to \underline{\mathrm{Spec}}_Y(f_*O_X) \to Y.$$

Denote $\underline{\mathrm{Spec}}_Y(f_*O_X)$ by Z. From the property of $\underline{\mathrm{Spec}}$, $O_Z = f_*O_X$, hence the first morphism f' has connected fibers(to show the properness of f', we claim that g is affine and hence separated, now f is proper implies f' proper). Finiteness of the second morphism g is from the fact that f_*O_X is a coherent O_Y-module ([6, II.5]). □



Now we turn to the two properties mentioned in 1.1, and then prove Theorem 1.4. The first one concerns the connected components of the fibers, which is straightforward by the Stein factorization. We use the same notations in Theorem 1.12.

Proposition 1.13. *There's a 1-1 correspondence between the connected components of the fiber $f^{-1}(y)$ and the maximal ideals of $(f_*O_X)_y$.*

Proof. From the factorization $f = g \circ f'$ in Theorem 1.12, the connected components of $f^{-1}(y)$ are 1-1 with the points in $g^{-1}(y)$, via $Z \mapsto f'(Z)$. But g is a finite map, hence points in $g^{-1}(y)$ correspond to the maximal ideals of $(g_*O_Z)_y$ where the latter equals to $(f_*O_X)_y$ by the construction of Z. □

In the following proposition we consider the points isolated in its fiber $f^{-1}(f(x))$. Use the notations as above.

Proposition 1.14. *The subset $T = \{x \in X : x \text{ isolated in its fiber}\}$ is open in X.*

Proof. Same as above, one verifies that x isolated in fiber if and only if $f'^{-1}(f'(x)) = x$, therefore we only need to consider f'. So it is sufficient to deal with the case $f = f'$ has connected fibers, $f_*O_X = O_Y$.

Assume $x \in T$, $f(x) = y$. Take affine neighbourhood $x \in U = \operatorname{Spec} B$, $y \in V = \operatorname{Spec} A$, $f(U) \subset V$. Now for f closed, $f(X-U)$ closed in Y, hence $f(U)$ open in V. Since the principal open sets $\{D(f) = \operatorname{Spec} A_f : f \in A\}$ form a basis of V, one can take $V_0 = \operatorname{Spec} A_s \subset f(U)$. We have $U_0 = f^{-1}(V_0) \subset U$, $U_0 = \operatorname{Spec} B_s$. By $f_*O_X = O_Y$, $f|_{U_0}$ is isomorphism, so $U_0 \subset T$ which implies T is open. □

Notice that the result in the previous proposition can be strengthen.

Proposition 1.15. *Assume $f_*O_X = O_Y$ in the previous proposition, then $f|_T$ is an open immersion.*

Proof. Straightforward by definition of T and $f_*O_X = O_Y$. □

With the preparations above, we can prove Theorem 1.4 easily, which factors a quasi-projective morphism into open immersion and finite morphism, when restrict it on open set T defined before.

Proof of Theorem 1.4. By definition, f can be factored as

$$X \hookrightarrow \mathbb{P}^n_Y \to Y.$$

The first morphism is open immersion so we only consider the second one. So we assume f is projective (hence proper) below. Let $f = g \circ f'$ be the Stein factorization, Proposition 1.14 shows $f'|_T$ is open immersion, so we get the conclusion. □

We know that an finite morphism has finite fibers, but the inverse is not true in general.

Example 1.16. Consider $X = \mathbb{A}^1_k - (0) = \operatorname{Spec} k[x, 1/x]$, $Y = \mathbb{A}^1_k = \operatorname{Spec} k[x]$, $f : X \to Y$ is the inclusion. Obviously all fibers of f are finite, but it is not finite for $k[x, 1/x]$ is not finite as module over $k[x]$. □

However, quasi-finite indeed implies finite in most common cases, for example when the morphism is projective. Using the Zariski's main theorem 1.4 one can deduce a relation between finite and quasi-finite morphism:

$$\boxed{\text{quasi-finite+proper=finite.}}$$

(Recall that a finite type morphism $f : X \to Y$ is quasi-finite at $x \in X$ if $\mathcal{O}_{X,x}$ is finite over $\kappa(x)$, or equivalent with fiber $f^{-1}(f(x))$ is finite, or x isolated in its fiber. ([5, 1.9]))

Theorem 1.17 (Chevalley)**.** *Let X, Y be Noetherian schemes, then morphism $f : X \to Y$ is finite if and only if quasi-finite and proper.*

Proof. The 'only if' part is easy: only need to show finite implies proper. Closedness is by the going-down property of finite homomorphsim, separatedness is because affine morphisms are separated.

Now we deduce the 'if' part by Theorem 1.4. Notice $T = X$ in this case so f' in the Stein factorization is open immersion, hence finite. And g is also finite, therefore f is finite. □

Proposition 1.18. *Let $f : X \to Y$ be a proper morphism, Y is locally Noetherian and the fiber X_y over y is finite(as set), then there exists open neighborhood V of y such that f is finite when restricted to $U = f^{-1}(V)$.*

Proof. Let $T \subset X$ be the open set of all the points that isolated in fiber, then $Z = f(X - T)$ is closed in Y for f proper. For $f^{-1}(y)$ is finite all points are in U, hence $y \in Y - Z$. Take open V that $y \in V \subset Y - Z$, $U = f^{-1}(V)$, $f|_U$ is quasi-finite hence finite by previous proposition. □

Remark 1.19. This proposition generalizes the so called generic finiteness which only deal with the situation that y is a generic point (of some irreducible component). □

Now we can prove the original form Theorem 1.1, to see 1.3 is indeed a generalization.

Proof of Theorem 1.1. Sufficient to prove the statement for $p_1^{-1}(P)$, i.e. $p_1^{-1}(P)$ is connected with dimension ≥ 1. Notice that $p_1 : \Gamma \to X$ is a birational projective morphism, we can use our Theorem 1.3 to conclude that $p_1^{-1}(P)$ is connected. If it has dimension 0, by the property of the dimension of fibers, one find open V contains P such that all points in V has fiber dimension 0, hence by theorem 1.16 p_1 is finite when restricted to $p_1^{-1}(V)$. But V is also normal, so p_1 is actually isomorphism when restricted to $p_1^{-1}(V)$. This indicates the birational map can be defined on V, which contradicts with P is fundamental. □

We end the section with another application, considering how a morphism behaves like isomorphism become isomorphism.

Proposition 1.20. *Let f be an birational quasi-finite morphism between Noetherian integral schemes. Suppose Y is normal and f is proper, then f is isomorphism.*

Proof. By Theorem 1.4, f is an open immersion. And f is also surjective for f has closed image by properness, so the image is just Y for Y is integral. □

Corollary 1.21. *Let $f : X \to Y$ be a birational morphism between projective curves with Y non-singular. If f is not trivial (i.e. image is a single point) then f is an isomorphism.*

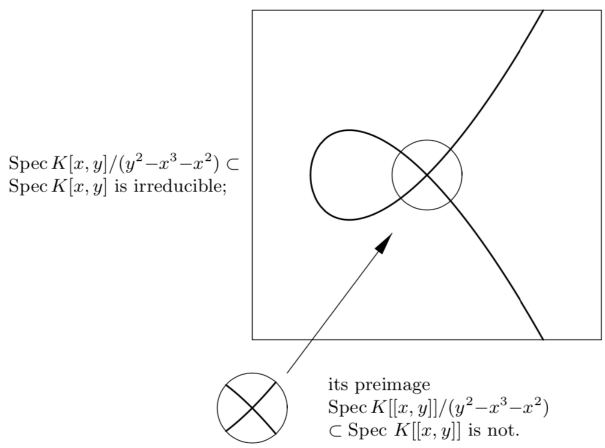
2 The Formal Theory

2.1 Examples for the Completion

One main defect of the Zariski topology is that it is too coarse and there're no enough open sets to detect the topological information of schemes.

Example 2.1.

- The Zariski topology on curve is the cofinite topology which has very big open sets.
- The cohomology of affine schemes at order > 0 is 0. However different affine schemes surely have very different structures. So the cohomology (in Zariski topology) can't reflect the geometrical information well.
- Consider the curve $C : y^2 = x^2(x+1)$, and assume the base field is the complex number $\mathbb{C}$. At the point $O = (0,0)$ it has two tangent lines $y = \pm x$. Visually if we restrict $C - O$ to a sufficient small neighborhood U of O, it will have 2 connected components(it is reasonable to expect C to be 'approximated' by $y = \pm x$ well near O).
  So one may expect that the local ring of the coordinate ring $A = \mathbb{C}[x,y]/(y^2 - x^3 - x^2)$ at O can be written as direct product of two rings, representing the two parts of the local functions on the two components. But it is clear the localization of an integral domain is again integral domain, hence not the product of two rings. This phenomenon reveals the defection of Zariski topology again: there're no enough opens so that one can't find a neighborhood U of O, such that $U - O$ splits into two components. □



So we will naturally expect some method to strengthen the Zariski topology to get rid of the contradictions with intuition. The last example above leads to the method we use here, i.e. the so called completion.

Example 2.2. Consider the last case in Example 2.1. In analytic case we study functions by its power series expansion locally, so we try to formally repeat the procedure here. In the ring $\mathbb{C}[[x,y]]$, if we set

$$u = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \cdots, \text{then } u^2 = 1 + x,$$

$$i.e.\ \sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \cdots,$$

hence $\widehat{A} = \mathbb{C}[[x,y]]/(y^2 - x^3 - x^2)$ will have the desired decomposition

$$\widehat{A} = \mathbb{C}[[x,y]]/(y+u)(y-u) = \mathbb{C}[[x,y]]/(y+u) \oplus \mathbb{C}[[x,y]]/(y-u).$$

In fact, the $\widehat{A}$ above is obtained by an algebraic procedure named *completion*. In the general setting, for ring A and ideal I, the *I-adic completion* is defined to be the limit

$$\widehat{A} = \varprojlim_{k \geq 1} A/I^k.$$

Here we take I to be the maximal ideal at O, i.e. the maximal ideal $(\overline{x}, \overline{y})$. □

In the previous discussion, we find to detect the geometry near a point, the completion indeed gives the correct information. Behind this example hides our main idea: rather than changing the topology of underlying space, we 'thickening' the functions over the space by completion.

Example 2.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring, M be finite generated module over A. The *local criterion of flatness* claims that if $M/\mathfrak{m}^k M$ are flat over $A/\mathfrak{m}^k$ for all $k > 0$, then M is flat over A ([8]). That is to say, when the all thickened fibers(over $A/\mathfrak{m}^k$) are flat, then M is flat. □

2.2 The Formal Function Theorem

In most of our situations, when considering morphism $X \to Y$, we can firstly take affine open subsets of Y and reduce to the situation that Y is affine. So we firstly assume the target is affine.

Let A be a Noetherian ring, I an ideal of A, scheme X proper over $\operatorname{Spec} A$, $\mathcal{F}$ a coherent sheaf on X. Let $f : X \to \operatorname{Spec} A$ be the structure morphism, we write $I\mathcal{F} = (f^* I^\sim)\mathcal{F}$; the product of an ideal sheaf and a coherent sheaf defines naturally. Now we have a natural morphism(in fact, this is what we called 'base change' morphism)

$$H^n(X, \mathcal{F})/I^k H^n(X, \mathcal{F}) \to H^n(X, \mathcal{F}/I^k\mathcal{F})$$

for any $k > 0$. And it is easy to see both side become a inverse system on the index k, hence take the inverse limit we get a canonical homomorphism

$$H^n(X, \mathcal{F})^\wedge \to \varprojlim H^n(X, \mathcal{F}/I^k\mathcal{F}).$$

The following theorem indicates it is an isomorphism.

Theorem 2.4 (formal function theorem)**.** *Notations as above. We have*

$$H^n(X, \mathcal{F})^\wedge \xrightarrow{\sim} \varprojlim H^n(X, \mathcal{F}/I^k\mathcal{F})$$

by the natural homomorphism constructed above.

We'll give a sketch of the proof in the section 2.5. For details, see [4, 2.5][2, 4.1].

Remark 2.5.

(i) Put $n = 0$ we get

$$\widehat{\Gamma(X, O_X)} \xrightarrow{\sim} \varprojlim \Gamma(X, O_X / I^k O_X).$$

The left side is the I-adic completion of the global sections(or 'holomorphic functions') of structure sheaf as A-module, and the right side is just the 'power series' or 'formal function'. That's the meaning of the name 'formal function theorem'.

(ii) This theorem says we get the same thing when completing the cohomology in two ways. Like the discussion in Section 2.1, take completion is somehow extending algebraic functions to analytic functions. Actually we will see in Section 2.4 that they all equals to $H^n(\widehat{X}, \widehat{\mathcal{F}})$. □

We restate the theorem in a global form, using higher direct image instead of cohomology.

Lemma 2.6. *Assume X be a Noetherian scheme with morphism $f : X \to \operatorname{Spec} A$, then for quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$R^n f_* \mathcal{F} \simeq H^n(X, \mathcal{F})^\sim.$$

Proof. See [6, III.8.5]. □

Theorem 2.7. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes, $\mathcal{F}$ is a coherent sheaf on X, $\mathcal{I}$ is coherent ideal sheaf of O_Y. Then we have canonical isomorphism*

$$u : \varprojlim (R^n f_* \mathcal{F} / \mathcal{I}^k R^n f_* \mathcal{F}) \xrightarrow{\sim} \varprojlim R^n f_* (\mathcal{F} / \mathcal{I}^k \mathcal{F}).$$

Proof. Notice on any topological space the inverse limit of sheaf is still a sheaf, and taking section commutes with taking inverse limit ([17, 009E]).
Firstly we have the canonical morphism $\varprojlim (R^n f_* \mathcal{F} / \mathcal{I}^k R^n f_* \mathcal{F}) \to \varprojlim R^n f_* (\mathcal{F} / \mathcal{I}^k \mathcal{F})$. By the previous lemma and Theorem 2.4,

$$\varprojlim (R^n f_* \mathcal{F} / \mathcal{I}^k R^n f_* \mathcal{F})(U) \to \varprojlim R^n f_* (\mathcal{F} / \mathcal{I}^k \mathcal{F})(U)$$

is an isomorphism for affine open U. Now the assertion follows by considering the kernel and cokernel of u. All sections over affine opens are 0, hence each stalk of them are 0, i.e. they are identical 0. □

Remark 2.8. We can interpret Theorem 2.7 as a 'base change theorem', since the canonical isomorphic morphism is just a limit of base change morphism. For étale cohomology and torsion sheaves, base change is exactly an isomorphism when requiring the proper condition, known as 'proper base change theorem'. But in coherent and Zariski cohomology settings, it is not an isomorphism in general. The solution is to pass to the 'formal fiber', as in theorem 2.4 and 2.7. □

Now we apply the theorem to a special case. Let $f : X \to Y$ be a proper morphism of Noetherian schemes, $\mathcal{F}$ is a coherent sheaf on X, y a point on Y. $X_n = X \times_Y \operatorname{Spec}(O_{Y,y} / \mathfrak{m}_{Y,y}^n)$ be thickened fibers, and $\mathcal{F}_k$ is the pullback of $\mathcal{F}$ along the canonical morphism $X_k \to X$. Applying Theorem 2.7 in the case $\mathcal{I} = \mathfrak{m}_{Y,y}$ we get

Theorem 2.9. *Notations as above. We have a natural isomorphism*

$$((R^n f_*\mathcal{F})_y)^\wedge \xrightarrow{\sim} \varprojlim H^n(X_k, \mathcal{F}_k).$$

where the completion is along the maximal ideal $\mathfrak{m}_{Y,y}$.

Proof. Let $i_k : \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^k) \to Y$, $j_k : X_k \to X$, $f_k : X_k \to \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ be the canonical morphisms.

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "j_n'"] & \mathrm{Spec}\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n \arrow[d, "i_n'"] \\
X' \arrow[r, "f'"] \arrow[d, "j"] & \mathrm{Spec}\mathcal{O}_{Y,y} \arrow[d, "i"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Notice that

$$\begin{aligned}
(R^n f_*\mathcal{F})_y &= H^0(\mathrm{Spec}\,\mathcal{O}_{Y,y}, i^*(R^n f_*\mathcal{F})) \\
&\simeq H^0(\mathrm{Spec}\,\mathcal{O}_{Y,y}, (R^n f'_*)(j^*\mathcal{F}))(\text{flat base change, Lemma 2.10}) \\
&\simeq H^n(X', j^*\mathcal{F})(\text{Leray spectral sequence for } f'),
\end{aligned}$$

and

$$H^n(X', j^*\mathcal{F})/\mathfrak{m}_{Y,y}^k H^n(X', j^*\mathcal{F}) \simeq H^n(X', j^*\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^k \simeq H^n(X_k, \mathcal{F}_k)$$

by flat base change again. Hence we have

$$\begin{aligned}
((R^n f_*\mathcal{F})_y)^\wedge &\simeq \varprojlim H^n(X', j^*\mathcal{F})/\mathfrak{m}_{Y,y}^k H^n(X', j^*\mathcal{F}) \\
&\simeq \varprojlim H^n(X, \mathcal{F}_k')(\text{Theorem 2.4}) \\
&\simeq \varprojlim H^n(X_k, \mathcal{F}_k).
\end{aligned}$$

□

Lemma 2.10 (flat base change)**.** *Let $f : X \to S$ be a quasi-compact separated morphism and $g : S' \to S$ be flat morphism, $X' = X \times_S S'$. $f' : X' \to S'$, $g' : X' \to X$ be the natural morphisms. $\mathcal{F}$ is a quasi-coherent sheaf on X. Then we have canonical isomorphism*

$$g^* R^n f_*\mathcal{F} \xrightarrow{\sim} (R^n f'_*)g'^*\mathcal{F}.$$

Proof. Reduce to affine case, the flatness condition implies f^* is locally exact. See [4, 2.4.10]. □

As an application of the theorem, we give the promised proof of the connectedness lemma.

Proof of the connectedness lemma. Let $f : X \to Y$ be a proper morphism between Noetherian schemes. Take any point y on Y, we show the fiber X_y is connected.

If X_y is not connected, say it can be written as $X_y = A \coprod B$ where A, B are closed non-empty sets in X_y, then $\Gamma(X_y, \mathcal{O}_{X_k}) = \Gamma(A, \mathcal{O}_{X_k}) \oplus \Gamma(B, \mathcal{O}_{X_k})$. Put $\mathcal{F} = \mathcal{O}_X$, $n = 0$ in Theorem

2.9, we have

$$\begin{aligned}\widehat{\mathcal{O}_{Y,y}} = ((f_*\mathcal{O}_X)_y)^\wedge &= \varprojlim \Gamma(X_y, \mathcal{O}_{X_k}) \\ &= \varprojlim(\Gamma(A, \mathcal{O}_{X_k}) \oplus \Gamma(B, \mathcal{O}_{X_k})) \\ &= \varprojlim \Gamma(A, \mathcal{O}_{X_k}) \oplus \varprojlim \Gamma(B, \mathcal{O}_{X_k}).\end{aligned}$$

Since A, B are non-empty, the direct summands in the right side are not 0. But the left hand side is the completion of a local ring with respect to its maximal ideal, hence still a local ring with maximal ideal $\widehat{\mathfrak{m}_{Y,y}}$. A local ring can't be written as direct sum of two non-trivial rings (consider the maximal ideal of these rings), so that's a contradiction. □

Remark 2.11.

(i) In fact, the result can be strengthened to $f_*\mathcal{O}_X = \mathcal{O}_Y$ implies the geometric fibers are connected, without any other efforts.

(ii) We can understand the condition $f_*\mathcal{O}_X = \mathcal{O}_Y$ as following: by definition, $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ so it is the 'functions on the preimage' or 'functions on the fiber' when U be a point. As an $\mathcal{O}_Y$ module, it should be 'bigger' than $\mathcal{O}_Y$ so when $f_*\mathcal{O}_X = \mathcal{O}_Y$, there're very few functions on the fiber. That indicates the fiber has some property relates with compactness (as we have no non-constant global function on compact complex manifold or projective variety). For example, when $X = \mathbb{A}^1_k \times \mathbb{A}^1_k = \mathbb{A}^2_k$, $Y = \mathbb{A}^1_k$ and f be the first projection, the condition doesn't hold but for $X = \mathbb{A}^1_k \times \mathbb{P}^1_k$, Y, f as above, the condition holds: in the second case, the fibers are projective, hence they have no non-constant global function. Generalize the example above a little, one can show that for projective morphism $f : X \to Y$ between Noetherian schemes, if all fiber are connected then $f_*\mathcal{O}_X = \mathcal{O}_Y$.

(iii) However, connectedness does not imply $f_*\mathcal{O}_X = \mathcal{O}_Y$ in general. Put $X = Y = \operatorname{Spec} A$, A be a character p perfect ring(for example, $A = \mathbb{F}_p$ is the finite field with p elements). Denote the Frobenius $a \mapsto a^p$ by $\varphi : A \to A$, $f : X \to X$ is the morphism induced by φ. Now $f_*\mathcal{O}_X = (A^{(p)})^\sim \neq \mathcal{O}_X$, where $A^{(p)}$ is the 'Frobenius twisted' A. As an A-module it has the structure of Abelian group same as A, with an action of A: $a.x = a^p x$ where $a \in A$, $x \in A^{(p)}$, and $(A^{(p)})^\sim$ is the coherent sheaf associated to $A^{(p)}$ ([Ha77, II.5]). But f is finite hence proper, and it is a homeomorphism on the underlying topological space of X so all the fibers are connected.

(iv) The example above suggests that we can get rid of the chaos caused by positive character by requiring the geometric fibers to be connected and reduced. We leave it to the readers to spell out the explicit statement and the proof. The idea is arguing by Stein factorization, as usual. □

Come back to Theorem 2.9 again. The right hand side is the limit of cohomology of fibers; by the result on cohomology dimension of Zariski cohomology ([6, III.2.10]), all cohomology vanish in degrees higher than $d = \max_y \dim X_y$, so we have the following corollary

Corollary 2.12. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes, $\mathcal{F}$ is a coherent sheaf on X, d defined as above. Then for any $i > d$,*

$$R^i f_*\mathcal{F} = 0.$$

In particular, when f is quasi-finite(for example, étale), f_* is exact.

Remark 2.13. In this case, $f^* = f^!$. See [5, 8.4] □

2.3 Formal Schemes

In Theorem 2.9 we have considered a direct system of schemes $\{X_k\}$, with same underlying spaces and thickened structure sheaves. Now we consider the general setting, to assume the underlying space be any closed set cut out by some ideal sheaf $\mathcal{I}$.

Definition 2.14. *Let X be a Noetherian scheme and $\mathcal{I}$ a coherent ideal sheaf. The* formal completion *of X with respect to $\mathcal{I}$ is defined by the ringed space $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ where*

$$\widehat{X} = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}),\ \ \mathcal{O}_{\widehat{X}} = \varprojlim \mathcal{O}_X/\mathcal{I}^k.$$

We have a canonical morphism of ringed spaces

$$(i, i^{\#}) : (\widehat{X}, \mathcal{O}_{\widehat{X}}) \to (X, \mathcal{O}_X).$$

And in fact it is a morphism of locally ringed space ([4, 1.5.12]).

Let X_n be the closed subscheme of X cut out by $\mathcal{I}^n$, we have

$$i_n : X_n \longrightarrow X, i_{nm} : X_n \longrightarrow X_m (n \le m),$$

$$\widehat{i}_n : (X_n, \mathcal{O}_{X_n}) \to (\widehat{X}, \mathcal{O}_{\widehat{X}}).$$

Summarised in the following commutative diagram.

$$\begin{tikzcd}
(X_n, \mathcal{O}_{X_n}) \arrow[r, "i_n"] \arrow[d, "i_{nm}"] \arrow[dr, "\widehat{i}_n"] & (X, \mathcal{O}_X) \arrow[d, "i"] \\
(X_m, \mathcal{O}_{X_m}) \arrow[r, "\widehat{i}_m"] & (\widehat{X}, \mathcal{O}_{\widehat{X}})
\end{tikzcd}$$

For $\mathcal{O}_X$-module $\mathcal{F}$, define the formal completion

$$\widehat{\mathcal{F}} = \varprojlim (\mathcal{F}/\mathcal{I}^k\mathcal{F})|_{\widehat{X}} = \varprojlim \widehat{i}_{n*} i_n^* \mathcal{F}.$$

Comparing with the canonical morphisms $\mathcal{F} \to i_{n*} i_n^* \mathcal{F}$, we get a canonical morphism

$$\mathcal{F} \to i_* \widehat{\mathcal{F}}.$$

Remark 2.15. For two coherent ideal sheaves $\mathcal{I}$ and $\mathcal{I}'$ which define the same closed subset, one find they actually define the same completion of X and $\mathcal{F}$. Without loss of generality, let $\mathcal{I}$ be the biggest ideal sheaf defining the closed set X_0, then $\mathcal{O}_X/\mathcal{I}$ is reduced. For any other ideal sheaf $\mathcal{I}'$ defining X_0, locally $\mathcal{I}$ will be the radical of $\mathcal{I}'$, so (using the Noetherian condition) there exists some n such that

$$\mathcal{I}' \subset \mathcal{I} \subset \mathcal{I}'^n.$$

Therefore the completions defined by $\mathcal{I}$ and $\mathcal{I}'$ are the same, that is, the procedure of taking completion only depends on the closed set one choose. □

Consider an inverse system of coherent sheaves $(\mathcal{F}_n, \varphi_{mn})$ where $\mathcal{F}_n$ is coherent $\mathcal{O}_{X_n}$-module and φ_{mn}: $\mathcal{F}_m \to i_{nm*}\mathcal{F}_n$ for any $n \le m$ satisfies that

- $\varphi_{nn} = \mathrm{Id}$ and $i_{mk*}(\varphi_{mn}) \circ \varphi_{km} = \varphi_{kn}$ for any $n \le m \le k$;
- φ_{mn} induces $i_{nm}^*\mathcal{F}_m \simeq \mathcal{F}_n$ for any $n \le m$.

An $\mathcal{O}_{\widehat{X}}$-module $\mathfrak{F}$ is called coherent if there exists such an inverse system that

$$\mathfrak{F} \simeq \varprojlim \widehat{i}_{n*}\mathcal{F}_n.$$

Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$, we naturally get an inverse system $(i_n^*\mathcal{F}, \varphi_{mn})$, hence

$$\widehat{\mathcal{F}} = \varprojlim \widehat{i}_{n*}i_n^*\mathcal{F}$$

is a coherent $\mathcal{O}_{\widehat{X}}$-module. Coherent $\mathcal{O}_{\widehat{X}}$-module of such form is called *algebraizable*, that is, it can be induced by a $\mathcal{O}_X$-module via its 'restrictions' on X_n.

Proposition 2.16. *An $\mathcal{O}_{\widehat{X}}$-module $\mathfrak{F}$ is coherent if each $\mathcal{F}_n = \widehat{i}_n^*\mathfrak{F}$ is coherent $\mathcal{O}_{X_n}$-module and the naturally defined inverse system gives*

$$\mathfrak{F} \simeq \varprojlim \mathcal{F}_n.$$

Proposition 2.17. *Notations as above. We have*

$$\mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^\wedge,$$

$$\mathrm{Hom}_{\mathcal{O}_{\widehat{X}}}(\mathfrak{F}, \mathfrak{G}) \simeq \varprojlim \mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n).$$

Proof. [4, 1.5.13, 1.5.19]. □

2.4 Comparison Theorems

We have mentioned before that $\widehat{X}$ is some kind of 'analytification', now let's make it clearer. For smooth variety X over $\mathbb{C}$, consider the analytic space X^{an} consisting of the closed points of X and with the usual Euclidean topology to be a complex manifold. X^{an} has the structure of ringed space with the sheaf of analytic functions $\mathcal{O}^{an}$. We have natural inclusion

$$i : X^{an} \to X.$$

Serre's *GAGA principles* ([13]) claim that

Theorem 2.18. *The functor*

$$i^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an})$$

is an equivalence of categories, and induces isomorphism on cohomology

$$H^n(X, \mathcal{F}) \xrightarrow{\sim} H^n(X^{an}, \mathcal{F}^{an}).$$

Where $\mathcal{F}^{an} = i^*\mathcal{F}$, $\mathrm{Coh}(X)$ is the category of coherent sheaves over X. It is clear that $\widehat{X}$ plays the role of X^{an} in our case, and we have similar comparison theorems here called *formal GAGA*.

Consider a scheme X proper over $\operatorname{Spec} A$ where A is Noetherian. Assume I is an ideal of A, $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, coherent ideal sheaf $\mathcal{I} = I^\sim\mathcal{O}_X$ is over $\mathcal{O}_X$. Let $\widehat{\mathcal{F}}$ and $\widehat{X}$ be the completions along $\mathcal{I}$, $\mathcal{F}_k = \mathcal{F}/\mathcal{I}^k\mathcal{F}$ be the pullback of $\mathcal{F}$ to the closed subscheme X_n defined by $\mathcal{I}^k$. Write $i : \widehat{X} \to X$ for the canonical morphism in this and the next section.

Theorem 2.19. *Notations as above. For any n, we have natural isomorphism*

$$H^n(\widehat{X},\widehat{\mathcal{F}}) \xrightarrow{\sim} \varprojlim H^n(X,\mathcal{F}_k).$$

We will also sketch the proof in section 2.5.

Remark 2.20. By the formal function theorem, we have

$$H^n(\widehat{X},\widehat{\mathcal{F}}) \xrightarrow{\sim} H^n(X,\mathcal{F})^\wedge.$$

□

Actually we have a comparison theorem just same as the complex case ([2, 5.1.4]).

Theorem 2.21. *Notations as above. Assume A is complete with respect to I, then the functor*

$$i^* : \mathrm{Coh}(X) \to \mathrm{Coh}(\widehat{X}),$$
$$\mathcal{F} \mapsto \widehat{\mathcal{F}}$$

is an equivalence of categories.

This is a consequence of the *existence theorem* Theorem 2.22 and Theorem 2.23.

2.5 Proof of 2.4 and 2.19

In this section, we fix integer $n \geq 0$ and write $H^n(\mathcal{F})$ for $H^n(X,\mathcal{F})$, for simplicity. Put $\mathcal{F}_k = F/I^k\mathcal{F}$. By the long exact sequence we have exact sequence

$$H^n(I^k\mathcal{F}) \to H^n(\mathcal{F}) \to H^n(\mathcal{F}_k) \to H^{n+1}(I^k\mathcal{F}) \to H^{n+1}(\mathcal{F}).$$

Set $R_n = \mathrm{Im}(H^n(I^k\mathcal{F}) \to H^n(\mathcal{F}))$, $Q_n = \mathrm{Im}(H^n(\mathcal{F}_k) \to H^{n+1}(I^k\mathcal{F})) = \mathrm{Ker}(H^{n+1}(I^k\mathcal{F}) \to H^{n+1}(\mathcal{F}))$. Hence there's exact sequence

$$0 \to H^n(\mathcal{F})/R_n \to H^n(\mathcal{F}_k) \to Q_n \to 0.$$

By the left exactness of $\varprojlim$, we have

$$0 \to \varprojlim H^n(\mathcal{F})/R_n \to \varprojlim H^n(\mathcal{F}_k) \to \varprojlim Q_n.$$

Therefore, it suffices to show the following

- Let $q_{m,n} : Q_n \to Q_m$ be the morphisms of the projective system, then there exists some N such that $q_{n,n+N} = 0$ for all n. This implies $\varprojlim Q_n = 0$.
- The projective system R_n defines the same topology of $H^n(\mathcal{F})$ with $I^m H^n(\mathcal{F})$. Hence the surjections

  $$H^n(\mathcal{F})/I^k H^n(\mathcal{F}) \to H^n(\mathcal{F})/R_k$$

  define an isomorphism

  $$H^n(\mathcal{F})^\wedge \xrightarrow{\sim} \varprojlim H^n(\mathcal{F})/R_k.$$

Combining above two points and the exact sequence, Theorem 2.4 was proved. For $H^n(\widehat{X},\widehat{\mathcal{F}})$, we have

$$H^n(\widehat{X},\widehat{\mathcal{F}}) = H^n(\widehat{X},\varprojlim \mathcal{F}_k) = H^n(X, i_* \varprojlim \mathcal{F}_k) = H^n(X, \varprojlim (i_k)_*\mathcal{F}_k).$$

For the projective system $(H^n(F_k), u_{m,n})$, we claim that there exists N such that for all m and $m' \geq m+N$, $\mathrm{Im} u_{m,m'} = \mathrm{Im} u_{m,m+N}$. This implies $H^n(X, \varprojlim (i_k)_*\mathcal{F}_k) = \varprojlim H^n(X,(i_k)_*\mathcal{F}_k)$ ([2, Ch.0, 13.3.1]), and Theorem 2.19 follows.

2.6 Grothendieck's Existence Theorem

We state the existence theorem now. In short, the theorem says if the base scheme is complete, then every inverse system of coherent sheaves is algebraizable.

Theorem 2.22. *Let A be a Noetherian ring complete with respect to ideal I, X is a scheme proper over* $\operatorname{Spec} A$*, $\mathcal{I} = I^{\sim}\mathcal{O}_X$ is a coherent ideal sheaf over X and $\widehat{X}$ is the completion along $\mathcal{I}$. Then for any coherent $\mathcal{O}_{\widehat{X}}$-module $\mathfrak{F}$, there exists unique coherent $\mathcal{O}_X$-module $\mathcal{F}$ up to isomorphism such that $\mathfrak{F} \simeq \widehat{\mathcal{F}}$, where the completion is taken along $\mathcal{I}$.*

Theorem 2.23. *Let A be a Noetherian ring complete with respect to ideal I, $A_0 = A/I$, $S = \operatorname{Spec} A$, $S_n = \operatorname{Spec} A/I^n$. Scheme X is proper over S, with $X_n = X \times_S S_n$. Then*

$$\operatorname{Hom}(X, Y) \xrightarrow{\sim} \varprojlim \operatorname{Hom}(X_n, Y_n).$$

Remark 2.24. We shall see a typical application of the existence theorem in the following Theorem 3.3. For the proof, one can refer to [4][7][2]. □

3 Application to Fundamental Groups

Recall that a local homomorphism $\varphi : A \to B$ of local rings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ is *unramified* if $\mathfrak{m}B = \mathfrak{n}$ and $A/\mathfrak{m} \subset B/\mathfrak{n}$ is a separable extension; is *étale* if it is unramifield and flat. A morphism $f : X \to Y$ is étale at $x \in X$ if the induced homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is étale; f is étale if it is étale at all $x \in X$. We call a finite étale morphism an *étale covering*. See [5, 2.2, 2.3][10, I] for other characterizations and properties.

For the definition and basic properties of étale fundamental group, see [5][10][14]. As the usual notation, we use $\pi_1(X, a)$ for the fundamental group of scheme X(where a is a fixed geometric point of X), $\pi_0(X)$ for the set of all connected components of X, $\operatorname{Et}(X)$ for the category of étale covering of X whose objects are finite étale morphisms to X.

Consider scheme X proper over $S = \operatorname{Spec} A$, where A be a Noetherian complete (or generally, Henselian) local ring. Let the closed and generic points of S be s and η, respectively, and $\overline{s}$, $\overline{\eta}$ be the corresponding geometric points. It is an algebraic analog of the following geometric model: X is a variety over $\mathbb{C}$, $S = \mathbb{D}$ is the unit disc in $\mathbb{C}$, $f : X \to S$ is surjective and proper(that is, the preimage of compact set is compact). The point 0 corresponds with s, and $\mathbb{D} - 0$ corresponds with η. When f is smooth (or a *submersion*, in the language of manifolds), Ehresmann's lemma states that for some neighborhood U of 0, $X_U = f^{-1}(U)$ is diffeomorphic with $U \times X_0$ where $X_0 = f^{-1}(0)$ is the fiber over 0. Therefore, when we take sufficient small ball U containing 0, geometric properties(for example π_0, π_1, $H^n, \cdots$) of X_U and X_0 should be similar.

So we may consider corresponding results for algebraic cases. In this section we compare the π_0 and π_1 of X and its special fiber X_s. The following Theorem 3.1 and 3.2 will be used in the proof of *proper base change theorem* of étale cohomology ([5][15]). As for the comparison of $H^n(X)$ and $H^n(X_s)$, the readers can refer to the discussions of *local acyclicity* and *smooth base change* in [5, 7.6, 7.7][16, Chapitre 1].

Theorem 3.1. *Let $f : X \to S = \operatorname{Spec} A$ be a proper morphism, A be a Noetherian Henselian local ring, and s is the closed point of S. Then the natural map*

$$\pi_0(X_s) \to \pi_0(X)$$

is bijective.

Proof. Let $X \xrightarrow{f'} S' \to S$ be the Stein factorization. Since S' is finite over S, we have S' also affine, $S' = \operatorname{Spec} A'$. By the property of Henselian local ring ([10][5]), $A' = \prod_i A_i$ is finite product of local A-algebras for A' is finite over A. Denote $\operatorname{Spec} A_i$ by S_i with closed point s_i, $X_i = X \times_S S_i$. Now $S = \coprod S_i$ hence $X = \coprod X_i$. We show X_i are exactly the components of X.

Let Y be a connected component of X_i. For f' proper, $f'(Y)$ must be closed in S' hence containing the closed point. Hence Y must intersect with the special fiber of X_i over S_i, thus contains the special fiber since it is connected by 1.12. Therefore, Y is the only component of X_i i.e. X_i connected. For $s \times_S S' = \coprod s_i$, X_i correspond to the connected components of X_s by

$$X_i \mapsto (X_i)_{s_i}.$$

□

Theorem 3.2. *Let $f : X \to S = \operatorname{Spec} A$ be a proper morphism, where A is a Noetherian complete local ring, S has closed point s. Then for any geometric point $\overline{x}$ of X_s, the natural map*

$$\pi_1(X_s, \overline{x}) \to \pi_1(X, \overline{x})$$

is an isomorphism.

We deal with the following general case which obviously generalizes Theorem 3.2.

Theorem 3.3. *Let A be a Noetherian ring complete with respect to ideal I, $A_0 = A/I$, $S = \operatorname{Spec} A$, $S_0 = \operatorname{Spec} A_0$. Scheme X is proper over S, with $X_0 = X \times_S S_0$. Then we have equivalence of category*

$$\operatorname{Et}(X) \simeq \operatorname{Et}(X_0)$$

by pullbacking the étale covering:

$$Y \mapsto Y \times_X X_0.$$

Proof. Assume $S_n = \operatorname{Spec} A/I^n$ be the thickenings of S_0 of higher order, $X_n = X \times_S S_n$ be the pullback to S_n.

- Firstly we show the functor is fully faithful, i.e.

$$\operatorname{Hom}(Y, Y') = \operatorname{Hom}(Y_0, Y_0'),$$

  where $Y_0 = Y \times_S S_0$, $Y_0' = Y' \times_S S_0$, Y, Y' are objects in $\operatorname{Et}(X)$. That's by the Theorem 2.23 and the following lemma.

  Lemma 3.4. *Let X be a scheme, X_0 be the closed subscheme with same underlying space with X. Then*

$$Y \to Y \times_X X_0$$

  defines the equivalence of category

$$\operatorname{Et}(X) \simeq \operatorname{Et}(X_0).$$

  Proof. See [5, 2.3.12]. □

- Now we show the functor is essentially surjective. The idea is once have an étale covering Y_0 of X_0, by Lemma 3.4 one find a compatible series of étale covers of X_n, then using the existence theorem to pass it to étale covering of X.
  Let $f_0 : Y_0 \to X_0$ be an étale covering. By the above lemma we can find étale coverings $f_n : Y_n \to X_n$ s.t.
  $$Y_n \simeq Y_{n+1} \times_{X_{n+1}} X_n.$$
  And f_n are proper morphisms for f_0 is proper. But étale morphism is quasi-finite, hence f_n are finite by Theorem 1.16.
  We sheafify the schemes Y_n using $\underline{\mathrm{Spec}}$. Let $\mathcal{A}_n = f_{n*}\mathcal{O}_{Y_n}$ be a coherent sheaf of $\mathcal{O}_{X_n}$-algebra (for finiteness), so $Y_n = \underline{\mathrm{Spec}}_{X_n} \mathcal{A}_n$. By $Y_n \simeq Y_{n+1} \times_{X_{n+1}} X_n$,
  $$\mathcal{A}_n \simeq \mathcal{A}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}.$$
  Now use Grothendieck's existence theorem to obtain coherent $\mathcal{O}_X$-module $\mathcal{A}$ such that
  $$\mathcal{A}_n = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}.$$
  And the morphisms defining the $\mathcal{O}_{X_n}$ algebra structures can be also 'algebraized' by morphisms of $\mathcal{O}_X$-module, hence $\mathcal{A}$ also have compatible $\mathcal{O}_X$-algebra structure. Now it is sufficient to show
  $$\underline{\mathrm{Spec}}_X \mathcal{A} \to X$$
  is étale. Only need to check the closed points. Let x be any closed point on X, for X is proper over S, its image s in S is also closed point. We use the following criterion.
  Lemma 3.5. *Assume k be a field, a finite dimensional k-algebra A is étale over k if and only if it is a finite direct product of separable extensions of k.*
  So $\mathcal{A}_0 \otimes \kappa(x)$ is a finite direct product of separable extension of $\kappa(x)$, where $\kappa(x)$ is the residue field at x. By $\mathcal{A}_0 = \mathcal{A} \otimes \mathcal{O}_{X_0}$ we have $\mathcal{A} \otimes \kappa(x) = \mathcal{A}_0 \otimes \kappa(x)$. If we can show $\mathcal{A}$ is locally free, this can be passed to $\mathcal{A}$. Therefore $\underline{\mathrm{Spec}}_X \mathcal{A} \to X$ is an étale covering inducing f_0, which implies the functor is essentially surjective.
- We know for modules projective is equivalent to locally free, hence it suffices to show the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, -)$ is exact.
  Notice that the functor
  $$\mathcal{G} \to \widehat{\mathcal{G}}$$
  is exact and faithful, where the completion is respect to the coherent ideal sheaf $\mathcal{I} = I^{\sim}\mathcal{O}_X$. So we pass to limit to using the properties of $\mathcal{A}_n$. By 2.17 and 2.19,
  $$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{G})^{\wedge} \simeq \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{A}}, \widehat{\mathcal{G}}) \simeq \varprojlim \mathcal{H}om_{\mathcal{O}_{X_n}}(\mathcal{A}_n, \mathcal{G}_n),$$
  where $\widehat{X}$ is the completion along $\mathcal{I}$, $\mathcal{G}_n$ is the pullback to X_n.

  By the definition of $\mathcal{A}_n$, they are all locally free. To show $\mathcal{H}om_{\mathcal{O}_{X_n}}(\mathcal{A}_n, \mathcal{G}_n)$ is exact for $\mathcal{G}$ we only need to check the stalks, hence only need to check on opens where $\mathcal{A}_n$ is free, and our assertion follows. □

Remark 3.6. Theorem 3.2 is still true when only assuming A is Noetherian Henselian local ring, using *Artin's approximation theorem.* □

For a proper morphism $f : X \to Y$ with connected geometric fibers and points $s, \eta \in Y$ such that $s \in \overline{\{\eta\}}$ (we call s the *specialization* of η), we consider the relation between $\pi_1(X_s)$ and $\pi_1(X_\eta)$. After some reduction, we can assume Y is Henselian local ring with $s = \overline{s}$ (i.e. with algebraic closed residue field at s) and s, η be the closed point and generic point as above. Let a, b be geometric points of $X_{\overline{s}}$ and $X_{\overline{\eta}}$, respectively. One have composition

$$sp : \pi_1(X_{\overline{\eta}}, b) \to \pi_1(X, b) \to \pi_1(X, a) \to \pi_1(X_s, a),$$

where the first map is natural map, the second is a chosen isomorphism, the third is the isomorphism in 3.2. The map sp is called the *specialization map.*

$$\begin{tikzcd} X_{\overline{\eta}} \arrow[r] \arrow[d] & X \arrow[d] & X_s \arrow[l] \arrow[d] \\ \overline{\eta} \arrow[r] & Y & s \arrow[l] \end{tikzcd}$$

Theorem 3.7. *Notations as above. If f is flat with geometrically reduced fibers, then the map sp is surjective.*

***Proof*.** It is sufficient to show that for connected étale cover $g : Z \to S$, $Z_{\overline{\eta}}$ is connected. By property of étale covering and for S affine, Z is also proper flat over S with geometrically reduced fibers, and by Theorem 3.1 the special fiber Z_s is connected. Therefore, $H^0(Z_s, O_{Z_s})$ is an Artinian local k-algebra and because $k = \overline{k}$ the residue field of $H^0(Z_s, O_{Z_s})$ is also k. But we also have Z_s is reduced, hence $H^0(Z_s, O_{Z_s}) = k$ in fact.

Now in order to show the connectedness of geometric fiber over $\overline{\eta}$, it suffices to show $g_*O_Z = O_S$ by connectedness lemma 1.2. By flat base change 2.10, the base change morphism $g_*O_Z \otimes k \to H^0(Z_s, O_{Z_s}) = k$ is isomorphism, so using Nakayama's lemma we have $g_*O_Z = O_S$, as desired. □

In general, if p is the characteristic of the residue field of the closed point, we should consider the 'prime-to-p' quotient of the fundamental group to get rid of the wildly ramified case.

Example 3.8. Consider the naive example $X = Y = \operatorname{Spec} \mathbb{Z}_p$, then $X_s = \operatorname{Spec} \mathbb{F}_p$ and $X_\eta = \operatorname{Spec} \mathbb{Q}_p, X_{\overline{\eta}} = \operatorname{Spec} \mathbb{Q}_p^{nr}$. We have $\pi_1^{(p)}(X_{\overline{\eta}}) = \operatorname{Gal}(\mathbb{Q}_p^{tr}/\mathbb{Q}_p^{nr}) = \prod_{\ell \neq p} \mathbb{Z}_\ell$, $\pi_1^{(p)}(X_s) = \left(\operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)\right)^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$.[1] □

We have the following *specialization theorem*:

Theorem 3.9. *Let $f : X \to Y$ be a smooth proper morphism of locally Noetherian schemes, with connected geometric fibers. Let $s, \eta \in Y$, $s \in \overline{\{\eta\}}$, p be the characteristic of the residue field at s.*

- *If $p = 0$, then sp is an isomorphism:*

$$\pi_1(X_{\overline{\eta}}, b) \xrightarrow{\sim} \pi_1(X_{\overline{s}}, a).$$

- *If $p > 0$, then sp induces an isomorphism on the largest prime to p quotient of fundamental groups*

$$\pi_1^{(p)}(X_{\overline{\eta}}, b) \xrightarrow{\sim} \pi_1^{(p)}(X_{\overline{s}}, a).$$

[1] $\mathbb{Q}_p^{nr}$ is the maximal unramified extension of $\mathbb{Q}_p$ and $\mathbb{Q}_p^{tr}$ is the maximal tamely ramified extension.

We will not prove this theorem here, since it needs some deep results like *Abhyankar's lemma* and *Nagata-Zariski purity.* One can refer [14, X].

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场论中的对称与拓扑

刘睿智

摘 要

本文将简要总结过去十年中我们对量子场论中的非微扰研究的进展, 主要关注对称, 拓扑以及反常. 限于笔者知识水平和本文篇幅, 很多细节无法仔细展开, 但笔者希望尽可能提供详细的参考文献, 供感兴趣的读者进一步研究. 尤其推荐这篇文章 [22] 和这本书 [16].

1 简介

量子场论是物理学中重要的一支, 它既是物理学的研究对象, 也是重要的工具. 在量子场论早期, 我们采用微扰方法来研究它, 这一时期最为卓著的成果大概要数 Schwinger 对于电子反常磁矩的计算, 其和实验数据有高度吻合. 大体来说, 微扰方法就是把我们希望计算的物理量表示成所谓耦合常数的幂级数形式, 然后逐阶计算 [27]. 和物理上取得的巨大成功成对比, 量子场论在数学上无法被严格化, 例如, 以上提到的幂级数几乎总是发散的, 物理学家往往满足于计算到前几阶然后和实验相对比; 同时, 数学家 Haag 证明, 量子场论中使用的所谓相互作用绘景也不能良好定义 [13]; 路径积分形式中的积分测度也无法严格构造. 在这段时期, 数学家往往感于物理学之非理性而拒绝深入这个新生的学科, 如 Hardy 就以“无用的纯数学”自满; 而物理学家希望突破数学严格性的束缚. 在这样的背景下, 当时的数学和物理几乎是各自发展, 但各自都卓有成效. 这段时间里, 所谓的数学物理研究往往是指偏微分方程, 群表示论这样相对古典的科目.

然而, 情况在 70 年代被打破, 物理一方, 人们发现很多现象中包含的耦合常数会很大, 上述微扰方法会彻底失效, 我们亟需一套全新的方法来研究这样的系统. 这段时期, 大量的拓扑概念进入到物理中, 并彻底改变了物理学的面貌, 这个思潮绵延至今. 这段时期的代表性工作如’t Hooft 和 Polyakov 引入的瞬子概念 [28]; Kosterlitz 和 Thouless 首先意识到涡旋这种拓扑缺陷可以造成体系的相变 [1, Chap 8].

还有另一种情况, 微扰方法是可行的, 但是计算会非常复杂, 而且掩盖了结果的本质. 例如在所谓手征反常的现象中, 为了证明手征反常的结果不会受到微扰的高阶项的修正, 物理学家做了十分复杂的计算. 然而物理学家 Albert Schwarz 首先意识到, 手征反常和优美的 Atiyah-Singer index 定理是一回事 [5], 后者是上世纪拓扑和几何领域的伟大成就. 而正因为这个结果是拓扑的, 它在高阶扰动下不会改变, 这就解释了前面提到的物理学家的计算为什么是正确的.

从此, 物理学和数学开始了一段新时期. 在古典时代, 数学和物理学的关系是简单的, 物理学产生问题, 物理学家转化为数学问题, 数学家解决, 当然古早的很多数学家也是物理学家. 然而, 新时代的数学物理却有些许不同. 我们的目的不仅仅出于实用目的希望解决物理问题, 还在于利用物理学的图像来理解数学结构. 在这些思潮的引领下, 80 年代迎来了数学物理的井喷. 其中最为人乐道的大概要数 Witten 一系列工作, 例如在 [29] 中, Witten 发展了所谓拓扑量子场论, 利用物理上路径积分的方法解释了 Jones 多项式, 是几何拓扑领域

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稿件信息: 本稿件于 4 月 6 日来稿, 5 月 31 日第一次修改, 5 月 31 日接受. 稿件编辑: 陈恒宇.

的伟大突破. 这部分内容, 数学背景的读者可以参考 [15], 物理背景的读者推荐阅读 Greg Moore 在 TASI 2019 的笔记和录课. 另一方面则是共形场论和弦论的大发展 [4][25].

然而, 尽管 Witten 的工作十分有趣, 其在数学上却有缺憾, 正如之前提到, Witten 的工作基于路径积分形式, 然而路径积分测度在数学上无法严格定义. 但数学家意识到, 他们并不真的需要定义路径积分, 他们仅仅需要描述路径积分的结果应该具有什么样的性质. 这种思路导致了函子场论 (functorial field theory) 及相应的公理的诞生 [8][22][16]. 我们一般称这套公理为 Atiyah-Segal 公理, 最早由 Segal 对共形场论提出 [26], 后来 Atiyah 对拓扑场论提出了类似的公理. 尔后, 数学家对量子场论的研究大都基于这套公理, 并且取得了卓越的进展. 这就是本文会主要谈论的事情.

简而言之, 对称和拓扑是物理学中的基本原理, 也是场论的非微扰研究的重要工具. 其中的重要概念数不胜数, 如对称破缺, 反常都是非微扰效应, 在场论中大有作用 [10], 它们都离不开对称与拓扑. 然而, 对称性这一概念在今天已经被极大地推广了 [17][9][19][6]. 而反常, 粗略地说对对称性的违背. 比如在经典力学中, 我们希望对称群保辛地作用在辛流形上, 但这并非总是可以做到的, 有时会有一些阻碍. 通过反常, 我们可以直接得到系统基态需要满足一些约束, 见 [10], 因此在物理上很受关注. 反常作为对称性的违背, 如果对称性被推广, 那么反常的适用范围也会越广.

很多现代物理理论的起点都是考虑所谓作用量, 它是一个积分型泛函, 其取值依赖于场位形, 真实的位形由它的极值点给出[1]. 作用量一般形如

$$S = \int_M \mathcal{L}(\phi, \mathrm{d}\phi, \dots) \tag{1.1}$$

其中 M 是时空流形, 一般是一个伪黎曼流形, 维数记为 n, $\mathcal{L}$ 是一个 n 形式称为拉氏量, $\phi : M \to \mathbb{R}$ 在物理上称为标量场[2]. 一个例子是

$$\mathcal{L} = \frac{1}{2}\mathrm{d}\phi \wedge \star\mathrm{d}\phi - \frac{m^2}{2}\phi^2\mathrm{vol}^n \tag{1.2}$$

称为自由标量场理论, 其中 m 称为质量, 第二项称为质量项, 其中的 vol^n 为体元. 该标量场在物理上可以描述无自旋粒子的产生、湮灭和相互作用, 当然, 为了简单, 我们这里没有加入相互作用, 如果加入 $\frac{\lambda}{6}\phi^3$ 就意味着我们加入一个三体碰撞的相互作用, 以此类推. 在经典物理中, 系统的运动方程由 S 关于 ϕ 的变分决定, 即考虑场的微小扰动 $\phi \to \phi + \delta\phi$, 那么泛函 S 的极值条件对应 $\delta S = 0$, 它将给出物理上关心的运动方程[3]. 这就是所谓最小作用量原理. 我们考虑 $m = 0$ 的情况, 那么这个计算十分容易, 考虑 $\phi \to \phi + \delta\phi$,

$$\delta S = \int_M \delta\phi \mathrm{d} \star \mathrm{d}\phi \tag{1.3}$$

这里我们假设了 M 是无边的, 因此分部积分可以扔掉边界项. 由 ϕ 的任意性可知 $\mathrm{d}\star\mathrm{d}\phi = 0$ 即 ϕ 满足拉普拉斯方程, 我们称之为 ϕ 满足的运动方程. 在物理上, 我们根据系统应该满足的对称性可以限制作用量的形式, 例如, 如果我们要求系统满足 $\phi \to -\phi$ 的 $\mathbb{Z}_2$ 对称性, 那么形如

$$S_{int} = \frac{\lambda}{3!}\int_M \phi^3\mathrm{vol}^n$$

这样的项就不能加入作用量中. 通过要求体系满足的对称性, 我们可以限制系统作用量的形式, 这是大部分近代物理理论的起点; 当然, 这种限制一般是很弱的, 所以通过对拉氏量增减一些项来产生新理论.

[1] 可以类比, 在力学中, 系统的平衡位置由势能的极值点给出, 作用量原理可以理解为这个观察的推广.

[2] 当然, 物理上所谓的场其实千奇百怪, 但它们大都是时空上某个丛的截面, 比如旋量场就是旋量丛的截面, 度规 (metric) 也是张量丛的截面. 为了简单, 我接下来只谈标量场, 即平凡线丛 $M \times \mathbb{R}$ 上的截面.

[3] 请注意, 此处 δ 一概表示微小变化量, 不表示外微分的伴随.

在量子场论中, 我们不仅要关心作用量, 更重要的是如下量, 称为“配分函数”

$$Z = \int D\phi e^{iS/\hbar} \tag{1.4}$$

其中积分遍及所有 $\phi : M \to \mathbb{R}$, $D\phi$ 为其上的“测度”, S 为作用量, 而 $\hbar$ 为约化普朗克常数. 在 $\hbar \to 0$ 的形式极限下, 上述积分可以用驻定相位近似 (saddle point approximation) 估计, 这就从量子层面“解释”了作用量原理.

一般地, 物理上所谓的场都可以看成两个空间 (多数情况下是流形) 之间的映射 f : $M \to N$, 其中 M 称为定义域 (domain), N 称为靶空间 (target space). 比如, N 可以取为 M 上的一个丛, 则 f 就是该丛的一个局部截面.

所谓反常就是我们预期系统具有一个对称性, 但是系统实际上并不具有这样的对称性的现象. 反常听上去像是个坏东西, 但并不是, 它在物理上十分有用. 一方面, 它对我们的理论提出了非平凡的限制, 例如弦论的临界维数 26 或者 10 就是从共形反常的消除得到的. 另一方面, 反常是一种非微扰效应, 可以帮助我们理解系统的非微扰性质, 见 [10].

2 量子力学与 Atiyah-Segal 公理

在量子力学中, 我们最关心的对象是所谓量子态, 量子力学假设, 所有量子态构成一个复希尔伯特空间 $\mathcal{H}$, 物理上记这样一个态矢量如 $|\psi\rangle$. 物理量是这个空间上的自伴算子. 具体而言, 考虑哈密顿量 H, 如果 $|\psi\rangle$ 恰好是本征值为 E 的本征态, 那么当系统处在这个量子态的时候, 我们测得能量值为 E. 对于一般的归一化的态 $|\psi\rangle = \sum_n a_n |n\rangle$, 其中 $|n\rangle$ 为 H 的本征值为 E_n 的正交归一化本征态. 那么对这个叠加态测量能量的结果是不确定的, 得到 E_n 的概率为 $|a_n|^2$.

关于量子力学, 另一件重要的事情是时间演化. 在经典力学中, 系统在相空间上演化的轨迹由 H 生成的哈密顿向量场决定. 这在量子力学中是类似的, 系统的时间演化是一个由 H 生成的酉变换. 时间演化算符

$$U(t_2; t_1) := e^{-iH(t_2 - t_1)} \tag{2.1}$$

从时间的角度看, 我们在每个特定的时间 t 定义一个希尔伯特空间 $\mathcal{H}_t$(对这里的简单情况, 它们全都自然同构), 然后量子力学相当于把一段时间区间 $[t_1, t_2]$ 变成一个算子 $U(t_2; t_1)$: $\mathcal{H}_{t_1} \to \mathcal{H}_{t_2}$.

从一维紧流形的分类看, 连通的一维紧流形要么是上述闭区间, 要么是 S^1, 我们自然要问, 定义在 S^1 上的量子力学是什么? 这在统计物理中自然地出现了, 此时量子力学最关心的是所谓配分函数, 即时间演化算符的迹

$$Z = \mathrm{tr}(e^{-\beta H}) \sim \left\langle \psi | e^{-\beta H} | \psi \right\rangle \tag{2.2}$$

形式上可以看成是 $t = -i\beta$ 的时间演化, 且注意初态和末态都是 $|\psi\rangle$, 因此我们相当于在时间上取了周期边界条件. 所以此时量子力学的效果是, 把 S^1 变成一个复数.

通过以上两个例子, 我们看出, 时空是否存在边界, 对量子力学的产出极端重要. (量子力学可以看成 0+1 维量子场论)

Atiyah-Segal 公理就是上述观察的进一步延伸. 其关键在于把上述对应上升成为一个函子, 然而在陈述这组公理之前, 我们需要简单介绍一下配边与配边范畴. 设 M, N 为 n 维流形, 如果存在一个 $(n+1)$-流形 W, 使得 $\partial W = M \coprod N$, 则称 M 和 N 为配边 (bordant) 的, 我们也可以要求一些附加结构, 比如要求所有这些流形取定一个定向, 得到定向配边的概念,

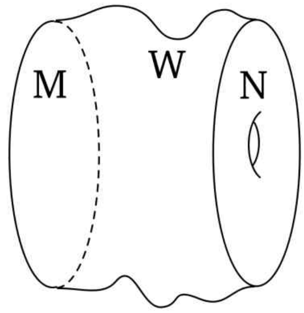

图 1.1: 配边, 图片引自 [21]

也可以要求他们具有自旋结构, 则得到自旋配边. 所谓配边范畴就是把 n 流形看成这个范畴的对象, 而上述的一个 N, 则看成 $M \to N$ 的态射, 可以验证这构成一个范畴, 恒等态射可以由柱 $M \times I, I = [0,1]$ 给出. 同时, 这个范畴是幺半范畴, 其张量积由流形的不交并给出. 我们记这个范畴为 $\mathrm{Bord}_n^{\bullet}$, 其中 $\bullet$ 可以填上 SO(表示定向配边), 也可以填 O(不定向配边), 填 spin(自旋配边) 以及其他选择.

A-S 公理的核心就是认为量子场论 F 是一个从配边范畴到希尔伯特空间范畴的幺半函子. 注意, 两个范畴中的张量积的单位分别是空的 n 维流形 $\emptyset^n$ 和 $\mathbb{C}$, 因此, 量子场论 F 满足

$$F(\emptyset^n) = \mathbb{C} \tag{2.3}$$

此外还有一些要求, 例如 $F(M^*) = F(M)^*$, 其中 M^* 表示 M 的定向反过来, $F(M)^*$ 则表示对偶空间.

我们再回头考察一下量子力学的例子, 闭区间 I 可以看成 $pt \to pt$, 记 $\mathcal{H} = F(pt)$, 则 $F(I) : \mathcal{H} \to \mathcal{H}$ 为线性变换. 另一方面 S^1 可以看成 $\emptyset^0 \to \emptyset^0$ 的态射, 因此 $F(S^1) : \mathbb{C} \to \mathbb{C}$, 其自然同构于 $\mathbb{C}$.

注 2.1. *我们评注一下为什么 A-S 公理仅适用于拓扑场论和共形场论*[4]*. 我们回忆, 为了让* $\mathrm{Bord}_n^{\bullet}$ *成为一个范畴, 我们要有恒等态射. 自然, 恒等态射会被映射成恒等算符. 恒等态射是用柱构造的, 但是我们对柱的具体长度并不关心 (显然也没有一个典则的长度可以选), 这对拓扑场论或者共形场论自然是无所谓的, 因为长度在这里不重要, 但对于一般的量子场论, 这个长度会重要, 因此我们缺少一个正则的恒等态射的选取. [16] 中采用了无穷小的配边作为恒等态射, 但可能会有分析上的困难.*

因此, 在 A-S 公理的框架下, 研究拓扑量子场论完全变成了一个范畴论和拓扑学问题. A-S 公理还有一个特点, 它根本不涉及场这个概念, 而是直指核心: 时空流形和希尔伯特空间.

3 边界上的新物理

然而, 后续研究表明, A-S 公理并不足以涵盖物理上所有有趣的情况 [14]. 为了说明进一步推广的必要性, 我们考虑两个特别的例子.

在过去 10 年中, 拓扑物态的研究获得了巨大的进展, 其中比较典型的一类叫做对称性保护拓扑序 (symmetry protected topological order, SPT)[30][12][5]. 所谓 SPT 物态, 如果在

[4]所谓拓扑场论就是指, 场论的最后计算出的可观测量, 只依赖于时空的拓扑结构, 而不依赖于比如度规这样的量; 所谓共形场论就是可观测量只依赖于时空的共形结构, 即度规的共形等价类, 由等价关系 $g \sim e^{2\sigma} g$ 决定, 其中 σ 为实光滑函数.

[5]其和拓扑序并不相同, 注意.

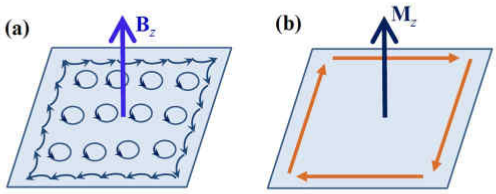

图 1.2: 拓扑绝缘体, 引自 [18]

通常的三维空间中, 样品不存在边界, 那么它的基态是唯一的, 且激发谱有能隙[6]. 所谓对称保护就是说, 如果不加任何对称性, 这样的态会完全平凡, 但加了一定的对称性之后, 它可以不平凡. 尤其, 当样品存在边界时, 对于不平凡的 SPT 物态, 边界上存在所谓't Hooft 反常, 其结果就是边界上存在无能隙、无损耗的边界激发模式或者对称性会自发破缺. 其中尤其典型的就是拓扑绝缘体, 其由电荷 $U(1)$ 对称性和时间反演对称性保护, 其边界上存在无损耗的传导电流.

物理学家和数学家假设这样的材料的低能性质 (即基态附近的性质, 由于这里基态唯一且无能隙, 这个场论的希尔伯特空间只有一个态) 可以用 $d+1$ 维拓扑量子场论描述, 根据 A-S 公理, 记该函子为 F. 根据我们的描述, 这个场论有特殊之处, 对于任何一个 d 维闭流形 M^d, $F(M^d)$ 的维数为 1, 这种场论称为可逆场论, 大致说是因为 1 维向量空间关于张量积是可逆的, 就像线丛构成 Picard 群一样. 然而, 前面说过, SPT 态最有趣的地方在于它的空间上的边界, 因此我们需要考虑量子场论对 $d-1$ 维流形, 即 $N^{d-1}=\partial M^d$ 的作用. 这完全符合物理实际, 但是超越了传统 A-S 公理的框架.

一种新框架基于所谓拓展量子场论 (extended field theory, ExFT)[16][2]. 我们先从 $d-1$ 维流形的角度来看, 根据配边的思想, d 维流形应该是 $d-1$ 流形之间的态射. 同理, $d+1$ 维流形是 d 维流形之间的态射. 这里会自然出现“态射的态射”, 这样, 一个自然的想法是引入高阶范畴论, 并且要求量子场论作为函子应该保持这个范畴的结构. 于是, 在代数侧, 我们要求范畴化向量空间. 因此, 我们在 $d-1$ 维流形上作用 F 的产物应该是一种 2− 向量空间 [23].

限于篇幅, 笔者不能在本文中展开 ExFT 的细节, nlab 有非常好的简介, 请参考 [24]. 不过还有另一种引入 ExFT 的动机, 我们可以继续说一说.

4 对称性及其推广

我们回忆一下诺特定理和 Ward 恒等式 [11]. 考虑一个 $d+1$ 维的场论, 其具有连续对称群 G, 则诺特定理表明存在守恒流 j^μ, 符合

$$\partial_\mu j^\mu = 0 \tag{4.1}$$

例如 $G=U(1)$ 的情况, 代入 $j^\mu=(\rho,\vec{j})$,$\partial_\mu=(\partial_t,\nabla)$, 则

$$\partial_t\rho+\nabla\cdot\vec{j}=0 \tag{4.2}$$

即为大家学过的电荷守恒方程. 其中 $\rho=j^0$ 为荷的密度, $\vec{j}$ 自然是流密度. 继续类比电磁学, 总电荷可以计算为

$$Q=\int_\Sigma j^0\mathrm{d}^3x \tag{4.3}$$

[6]就像在真空中激发一个质量为 m 的粒子至少需要 mc^2 的能量一般, 这种情况称为激发谱有能隙. 如果是光子这种静质量为 0 的情况则称为没有能隙.

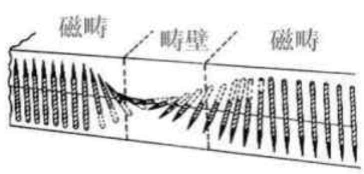

图 1.3: 畴壁, 引自 [7]

在物理中有一个非常深刻的洞见, 即守恒荷同时生成了变换本身. 为了稍微解释一下这个命题, 我们知道, 角动量是空间转动的守恒荷. 例如 $L_z := xp_y - yp_x$, 我们考虑对易子 $[x, p_x] = [y, p_y] = \mathrm{i}$, 其余对易子 $[x, y] = [p_x, p_y] = [x, p_y] = [y, p_x] = 0$. 则 $[L_z, x] = \mathrm{i}y$, $[L_z, y] = -\mathrm{i}x$. 我们再观察一下绕 z 轴转动下 x, y 的变换

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \theta \begin{pmatrix} -y \\ x \end{pmatrix}, \theta \to 0 \tag{4.4}$$

注意第二项刚好可以写成

$$\mathrm{i}[L_z, \begin{pmatrix} x \\ y \end{pmatrix}] \tag{4.5}$$

因此, 我们说, 角动量 L_z 生成了绕 z 轴的转动. 这个原理可以推广到很多情况. 它实质上表明诺特定理的逆定理也成立: 给定一个守恒荷也可以构造相应对称变换. 这个事实在辛几何的语境下极其显然, 利用动量映射生成哈密顿向量场即可, 我们不再展开.

推广到场论中, 比如 ϕ 是一个场算符, Q 是某个对称性的守恒荷算符, 那么如果

$$[Q, \phi] = \mathrm{i}q\phi \tag{4.6}$$

我们称 ϕ 携带有荷 q, 在有限大的变换下会形如 $\phi \to \mathrm{e}^{\mathrm{i}q\theta}\phi$, 其中 θ 是一个变换的参数.

请注意我们的讨论都是对一个特定的时间 t 而言的, 比如 Q 本身就是定义在一个等时面上面的. 从时空的角度看, 我们选出了一个等时面, 给它附加了一个对称变换 (由 Q 生成), 当我们的场算符或者别的局域算子跨过这个等时面的时候, 就会经历一个对称变换. 等时面即一个余维数为 1 的子流形, 从时空的角度, 我们把这样的结构称为畴壁 (domain wall).

畴壁这个名字源于大家对量子磁性的研究, 见图 3, 记箭头指向 (即自旋变量) 向上 (下) 为 +1(-1), 那么跨过畴壁的时候自旋变量 $s \to -s$. 这里只考虑空间, 不计时间维, 那么畴壁是 $1 - 1 = 0$ 维流形, 即一些点. 在图上看起来并非如此, 但是注意畴壁的尺寸相比样品很小, 可以忽略不计, 因此我们上面的说法仍然是成立的.

以上是传统的对称性的讨论, 对称这一概念在今天被极大地推广了. 我们举所谓高阶形式对称性 (higher-form symmetry) 为例 [17].

例 4.1. *考虑自由电磁场拉氏量 $\mathcal{L} = -\frac{1}{4}F \wedge \star F$, 其中 F 一个 $U(1)$ 主丛的曲率 2-形式, $F = \mathrm{d}A$, 其中 A 为联络 1-形式. 考虑如下变换 $A \to A + \lambda$, 其中 λ 为闭 1-形式, 此时 F 不受影响, 因此拉氏量不会改变. 然而这并不是一个规范变换, 因为物理上的可观测量 holonomy(物理上叫 Wilson loop)*

$$\mathrm{e}^{\mathrm{i}\oint_\gamma A} \to \mathrm{e}^{\mathrm{i}\oint_\gamma A}\mathrm{e}^{\mathrm{i}\oint_\gamma \lambda} \tag{4.7}$$

发生了变化. 但我们知道, 规范变换是不能改变可观测量的. 因此, 这并不是规范变换, 而是一个对称变换, 其变换参数不是一个标量, 而是一个闭 1-形式 λ, 我们因此称之为 1-形式对

称性. 它的“守恒流”$J = \star F$ 是一个 2-形式, 而不是 1-形式. 守恒方程为

$$d \star J = 0 \tag{4.8}$$

这其实是 Bianchi 恒等式, 它的守恒荷就不能定义在一个等时面上了, 只能定义在一个 2 维子流形上

$$Q = \frac{1}{2\pi}\int_\Sigma \star J = \frac{1}{2\pi}\int_\Sigma F \tag{4.9}$$

(归一化因子 2π 是为了让 Q 为整数) 这恰好是总磁荷数, 或者叫第一陈数. 1-形式对称性在一个余维数为 2 的子流形上赋予一个对称变换, 一般地, $n-$ 形式对称性在余维数为 n 的子流形上赋予对称变换.

另一方面, 我们上面说了, holonomy 是物理上重要的可观测量, 对于 Chern-Simons 理论, 它甚至是最主要的可观测量 [29], 因此我们也需要在理论中容许它们. 所有这类理论中的低维流形 (相对于时空), 都被称为缺陷 (defect). 著名数学物理学家 G.Moore 说:“Theories without defects are defective.(没有缺陷的理论是不完整的)”或者可以说, 场论的真空是土地, 缺陷是庄稼, 土地怎么样得看庄稼长得如何. 如果我们要考虑这些缺陷的话, 那么必须在理论中允许更低维的流形, 这是另一个通向 ExFT 的动机.

值得一提的是, 虽然 n-形式对称性推广了对称参数的取法 (准确说是改变了序参量), 但是这种对称性代数上仍然由群及其表示描述. 还有别的一些广义对称性, 它们由别的代数结构描述, 比如 n-group[9], 或者范畴 [6].

5 结语

有关量子场论的当代研究是无论如何也说不完的, 限于篇幅和笔者知识, 我们不得不在此打住. 但是能见证我们对于对称性的观念的天翻地覆的变革, 我认为是一件非常令人激动的事情. 这件事也是由一代又一代数学家和物理学家共同努力才得以实现, 从 [14] 首先讨论高阶范畴在物理上的应用, 到 n-group 和高阶规范理论 [3], 到 [20], 再到 [17] 最终在凝聚态上广为人知, 其中的过程实在不能蕴于一两句话. 希望以后对此感兴趣的同学能越来越多.

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Calabi's Program and Stability Theory

Junhao Tian

Abstract

This lecture notes give an introduction to Calabi's program about finding "canonical" metric on a given manifold. We will begin with Yau's famous work on Calabi Conjecture. We outline GIT analysis of stability of bundles and variety, and their connections to the theory of Kälher-Einstein(KE) metrics and constant scalar curvature Kähler(cscK) Metrics, also briefly sketch out some recent developments.

Acknowledgements. I am very grateful to all my teachers and my friends for their constant encouragement during my undergraduate. Among them I should mention Professors Bing Wang, Song Sun, Xi Zhang, Mao Sheng, Yongqi Liang, Zuoqin Wang, Bin Xu and my friend Dashen Yan. I have learnt Geometric Analysis from my supervisor Bing Wang over many years, his help and his encouragement can hardly be exaggerated. This paper is nothing but the notes about what I have read during my Undergraduate Research Program. I am also very grateful to my classmate Dashen Yan for many helpful conversations on Kähler-Einstein metric.

1 Introduction

1.1 What is a "nice" metric?

Geometric analysts always want to find a "nice" metric on a given manifold, which seems to be canonical. The definition of "nice" might be a subtle problem. A natural idea is that it makes some common functional, such as scalar curvature functional, achieve its minimum.

Scalar curvature is a higher dimensional analogue of Gauss curvature. Yamabe set up a problem to ask if one can find a metric such that its scalar curvature is equal to a given smooth function, moreover, in a given conformal class. In [113], he attempted to show that any Riemannian structure on a compact manifold of dimension not less than 3 could be pointwise conformally deformed to one of constant scalar curvature. Trudinger [106] pointed out a serious gap in Yamabe's proof, and the assertion is in doubt. Kazdan and Warner [56] had proved that, as long as the function is negative somewhere, there is a metric whose scalar curvature coincide with the given function. With this in mind, finding a metric whose scalar curvature is positive at every point sounds easy, since it is only a scalar inequality on the entire metric. However, there is a topological obstruction to the existence of metrics with positive scalar curvature.

Actually, some manifold may not admit a metric with positive scalar curvature everywhere. For example, the torus $\mathbb{T}^n$(cf. [84, 51]). In dimension 2, the problem of Gaussian

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稿件信息: 2021 年 12 月 21 日来稿, 2022 年 3 月 20 日第一次修改, 3 月 21 日接受. 稿件编辑: 章俊彦, 陈恒宇.

curvature on 2-manifolds was studied [57]. The key to our study of Gaussian curvatures was the Gauss-Bonnet theorem which imposes sign restrictions on the Gaussian curvatures of compact 2-manifolds depending on the Euler characteristic. There is also a topological implication of scalar curvature which introduces an obstruction to positive scalar curvature for certain special manifolds. Lichnerowicz has shown [61] that if the scalar curvature is nonnegative, but not identically zero, on a compact even-dimensional spin manifold, then there are no harmonic spinors. From this fact, using the Atiyah-Singer index theorem [3] he concluded that the Hirzebruch $\hat{A}$ genus of such a manifold must be zero. Thus, one cannot find a metric with nonnegative scalar curvature, except possibly identically zero, on a compact spin manifold whose $\hat{A}$ genus is not zero. Examples of such manifolds arise in the theory of spin cobordism, see [69].

There is a generalization of $\hat{A}$ that could completely characterize when a (simply-connected, spin) manifold admits a metric of positive scalar curvature. It is usually denoted by $\alpha(M)$, and was first introduced in 1974 by Hitchen [52] who showed that if there is a metric with positive scalar curvature, then $\alpha(M) = 0$. The converse was proven by Stolz in 1990 [87]. For our purposes, the most important property is that it is nonzero only in dimensions $n = 0, 1, 2, 4$ mod 8. (Here we assume $n > 4$.) So there is a possible obstruction to the positive scalar curvature only in these dimensions. Even in these dimensions, this obstruction only appears in the case of spin manifolds: If M does not admit spinors, then it always admits a metric of positive scalar curvature [50].

In view of Yamabe's problem, a metric is considered to be "nice" if it has constant scalar curvature. It seems to be special, but such a constant may not be arbitrary. Yabame invariant $\mu(g)$ is a conformal invariance that characterizes such a constant: The nontrivial solution of its Euler-Lagrange equation

$$-4\frac{n-1}{n-2}\Delta_g f + S(g)f = \mu(g)f^{\frac{n+2}{n-2}} \tag{1.1}$$

whose existence is guaranteed by the solution of Yamabe problem [83], gives rise to the so-called Yamabe metric $f^{\frac{4}{n-2}}g$, which has constant scalar curvature $\mu(g)$.

1.2 Calabi's program

In 1950s, Calabi first proposed to study the constant scalar curvature Kähler (cscK) metric problems. His goal is to find the best canonical metric in each given Kähler class [15, 16], which introduced the study of a 4th order, fully nonlinear PDE(partial differential equation). The related PDE is very difficult, for one cannot use the maximal principle or derive an appropriate estimates of the metric from the bound of the scalar curvature. When the first Chern class has a definite sign (positive, negative or zero), the cscK metric in the suitable multiple of the first Chern class reduces to a Kähler-Einstein metrics, which is the core of this research field for the last a few decades. Mathematicians put great efforts and developed lots of techniques and finally led to the final resolution of this difficult problem.

In 1958, E. Calabi proved the fundamental C^3-estimate for Monge-Ampère equation [14] which later played a crucial role in Yau's seminal resolution of Calabi conjecture [114] in 1976 when the first Chern class is either negative or zero (In the negative case, T. Aubin has an independent proof [4]) . This work of Yau is so influential that generations of experts in Kähler geometry largely followed the same route: Securing a C^0 estimate first, then move on to obtain C^2, C^3 estimates etc. In the case of positive first Chern class, Gang Tian proved Calabi conjecture in 1989 [94] for the Fano surfaces when the automorphism group is reductive. It is well known that there are obstructions to the existence of KE metrics in Fano manifolds; around 1980s, Yau proposed a conjecture which linked the existence of Kähler Einstein metrics with the stability of underlying tangent bundles. In 1997, Gang Tian introduced the so-called K-stability (via special degeneration) and showed that the existence of Kähler-Einstein metric necessarily implies the K-stability of the underlying polarization through special degeneration [99]. In 2002, S. K. Donaldson reformulated it into a notion of the algebraic K-stability [40]. This conjecture was settled in 2012 through a series of work [29, 30, 31], which is quite involved as it sits at the intersection of several different subjects: algebraic geometry, several complex variables, geometry analysis and metric differential geometry etc.

With the existence problem of Kähler-Einstein metric settled eventually, perhaps it is time to discuss how to tackle Calabi's original problem in full generality.

Conjecture 1.1 (Yau-Tian-Donaldson)**.** *Polarized Kähler manifold (M,L) is K-stable if and only if there exist a cscK metric in* $c_1(L)$*.*

In 2015, X. X. Chen proposed a "new" continuity path in a given Kähler class to solve the cscK metric problem [24]. Also in 2018, X. X. Chen and J. R. Cheng derived the a priori estimates for the constant scalar curvature Kähler metrics on a compact Kähler manifold, and proved Donaldson's conjecture on the equivalence between geodesic stability and existence of cscK when $Aut_0(M,J) \neq 0$ [25, 26, 27]. This deep result generalizes Tian Gang's Properness theorem, and the Mabuchi energy is proper if and only if there is a metric of constant scalar curvature in the class $[\omega]$. On the other hand, Sean Paul gives a complete description of the behavior of the Mabuchi energy along all degenerations. Under the assumption that $Aut(M,J)$ is finite, this gives the equivalence between the analytic stability and the algebraic stability.

Theorem 1.1. *Let* (X,L) *be an arbitrary polarized manifold. Assume that* $Aut(M,J)$ *is finite. Then* (X,L) *is asymptotically K-stable if and only if there is a constant scalar curvature metric in* $c_1(L)$*.*

The most important idea is to identify the norms conformally equivalent to the standard L^2 norms on polynomials. Since the conformal factors are continuous, they are bounded due to the compactness. The conclusion was that the Mabuchi energy is almost the distance between the orbits in Hilb, that is, the distance in the usual Fubini Study metric induced by L^2 up to some (unknown) error that depends on the degree of the embedding. Based on the works by Bismut, Gillet, and Soulé [7, 8, 9], Paul [74] recently found a more sophisticated path to the relationship between the Mabuchi energy restricted to the Bergman metrics and the resultant and hyperdiscriminant of the subvariety. This revealed that the error was in fact the difference between the L^2 norm and another well-known L^0 norm, i.e. the Mahler measure. The boundedness of the error, initially attributed to compactness, is just an expression of the fact that these norms are comparable. The outcome is that the norm on the space of polynomials, which connects the Mabuchi energy to stability of the pair (R,Δ), is exactly given by the Mahler measure. Now the asymptotic stability and the global bounds of the K-energy maps immediately follow from Tian's Thesis [95].

There is also another approach, called Kähler-Ricci flow, to the study of the existence of Kähler-Einstein metrics on Fano manifolds. In general, two key ingredients are needed, namely the partial C^0-estimate and the construction of a de-stabilizing test configuration. The first is analytic and the second is algebraic in nature. For the partial C^0-estimate, it is proved by Székelyhidi [90] for the classical Aubin-Yau continuity path, by adapting the results of [44, 30, 31]. For the approach using Ricci flow, this was proved by Chen-Wang [110] in dimension two, Tian-Zhang [104] in dimension three, and Chen-Wang [111] in all dimensions as a consequence of the resolution of the Hamilton-Tian conjecture. We note that these results together with the work of Sean Paul [74, 75, 76] already implied that on a Fano manifold without non-trivial holomorphic vector fields, the existence of a Kähler-Einstein metric is equivalent to the notion of stability defined by Paul. As for the second ingredient, Datar and Szëkelyhidi [107] adapted the results of [31] to the Aubin-Yau continuity path, which gives a new proof of the theorem of Chen-Donaldson-Sun. Chen-Wang [112] gave another proof using the Ricci flow, which means that technically they will address the issue of constructing a de-stabilizing test configuration. Notice that this cannot be naively adapted from [31] and requires new strategy to understand the relation between the asymptotic behavior of the Kähler-Ricci flow and algebraic geometry. Their work is motivated by [82] which studies the tangent cones of non-collapsed Kähler-Einstein limit spaces.

When considering about cscK metrics, there is a flow, called Calabi flow, which is supposed to be used to get cscK metric. Motived by Donaldson's theorem that links the balanced embeddings to the metrics with constant scalar curvature [39], Joel Fine [47] proved the parabolic analogue, balancing flow, which could approximate the Calabi flow using Donaldson's techniques [42] with an asymptotic result of Liu and Ma [62]. But I guess the problem about the long time existence of Calabi flow is still open. For the limitation of space, we will not discuss the geometry flow method in this article.

2 Calabi Conjecture

Let us start with Calabi's initial conjecture about the existence of certain Riemannian metrics on complex manifolds with given Ricci curvature. Let M be a Kähler manifold with Kähler metic $g = \Sigma g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$ and fundamental form ω of g, ρ is the Ricci form of ω. It is well known that $\frac{1}{2\pi}\rho$ represents the first Chern class of M. Calabi made the following conjecture in 1954 and that was proved by Shing-Tung Yau in 1977.

Conjecture 2.1 (Calabi)**.** *If a 1-1 form $\frac{1}{2\pi}\widetilde{\rho}$ represents a first Chern class, then there is a unique Kähler form $\widetilde{\omega} \in [\omega] \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$ that lies in the same de Rham cohomology class of ω such that its Ricci curature is $\widetilde{\rho}$.*

Remark 2.1. *Yau received the Fields Medal in 1982 in part for this proof. Calabi transformed the conjecture into a nonlinear PDE problem, called complex Monge-Ampère equation, and showed that this equation has at most one solution and thus established the uniqueness of the required Kähler metric. Yau proved the Calabi conjecture by constructing a solution of this equation using the continuity method in the middle of 1976. This involves first solving an easier equation, and then showing that a solution to the easy equation can be continuously deformed to a solution of the hard equation. The hardest part of Yau's solution* *is the estimates up to second and third orders for the derivatives of solutions.*

Proof of Calabi Conjecture by the continuous method. Let $\widetilde{\rho}$ is a 1-1 form such $\frac{1}{2\pi}\widetilde{\rho}$ represents the Chern class, it was well known that $\widetilde{\rho}-\rho$ is exact, then by $\partial\bar{\partial}$-lemma: $\exists\ F \in C^\infty(M,\mathbb{R})$, called Ricci Potential function, such that

$$\widetilde{\rho}-\rho = -\sqrt{-1}\partial\bar{\partial}F. \tag{2.1}$$

In the local coordinates, we can write this quotient as a volume form

$$\widetilde{\rho}-\rho = -\sqrt{-1}\partial\bar{\partial}\log\frac{\det\widetilde{g}_{i\bar{j}}}{\det g_{i\bar{j}}}. \tag{2.2}$$

If there exists $\widetilde{\omega} \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$ that lies in the same de Rham cohomology class of ω, by $\partial\bar{\partial}$-lemma, $\exists\ \varphi \in C^\infty(M,\mathbb{R})$ such that

$$\widetilde{g}_{i\bar{j}} = g_{i\bar{j}} + \partial_{i\bar{j}}\varphi. \tag{2.3}$$

By normalizing F, we now transfer the Calabi conjecture to the question that the equation below has a unique solution or not.

$$\frac{\det\left(g_{i\bar{j}} + \partial_{i\bar{j}}\varphi\right)}{\det g_{i\bar{j}}} = e^F. \tag{2.4}$$

This is Monge-Ampère equation. The uniqueness part is simple by using the maximal principle. See Theorem 3.2 for details.

Now we claim that the equation above has a solution $\varphi \in C^\infty(M)$. we are going to prove this claim using continuous method. Consider the set

$$S = \left\{t \in [0,1] \,\middle|\, \text{The equation } \tfrac{\det\left(g_{i\bar{j}}+\varphi_{i\bar{j}}\right)}{\det g_{i\bar{j}}} = Vol(M)\tfrac{e^{tF}}{\int_M e^{tF}} \text{ has a solution in } C^\infty(M)\right\}.$$

Obviously $0 \in S$ for $\varphi = 0$ is a solution. If we can show the set S is both open and closed, this will imply that $1 \in S$.

To see S is open, we consider a map $G : A \to B$, where

$$\begin{aligned}
A &:= \left\{\varphi \in C^\infty(M) \,\middle|\, \sum_{i,j}(g_{i\bar{j}} + \varphi_{i\bar{j}})dz^i \otimes d\bar{z}^j \text{ defines a metric on M and } \int_M \varphi = 0\right\},\\
B &:= \{f \in C^\infty(M) | \int_M f = Vol(M)\ \},\\
G(\varphi) &:= \det(g_{i\bar{j}} + \varphi_{i\bar{j}})\det(g_{i\bar{j}})^{-1},
\end{aligned}$$

and it is not difficult to compute the differential of G at the point φ_0

$$\tfrac{d}{dt}\Big|_{t=0} G(\varphi_0 + t\varphi) = \det(g_{i\bar{j}} + \varphi_{0,i\bar{j}})\det(g_{i\bar{j}})^{-1}\Delta_{\varphi_0}\varphi.$$

By Hodge decomposition theorem, $\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}})\det(g_{i\bar{j}})^{-1}\Delta_{\varphi_0}\varphi = f$ has a unique solution $\varphi \in C^\infty(M)$ if we require $\int_M f = \int_M \varphi = 0$. Hence the differential of G at φ_0 is invertible, which means G maps an open neighborhood of φ_0 to an open neighborhood of $G(\varphi_0)$. This proves S is open.

We also need to prove S is closed. Let $\{t_q\}_{q=1}^{\infty}$ is a sequence in S where $\lim_{q\to\infty} t_q = T$ then we have a sequence $\varphi_q \in S$ such that

$$\frac{\det(g_{i\bar{j}} + \varphi_{q,i\bar{j}})}{\det g_{i\bar{j}}} = Vol(M)\frac{e^{t_q F}}{\int_M e^{tqF}}. \tag{2.5}$$

We want to show that the sequence converges in some sense, maybe up to subsequence. To see this, we need the a priori estimates.

C^0-estimate: We use the method of Nash-Moser iteration [17]. Let us introduce J energy function

$$J(\varphi) \coloneqq \int \varphi([\omega_\varphi^n] - [\omega^n]). \tag{2.6}$$

Denote $v = \varphi - \sup\varphi - 1$, we can prove

$$J\left(\frac{(-v)^{p-1}}{p-1}\right) \geqslant \frac{1}{n}\int_M \frac{4}{p^2}|\nabla(-v)^{p/2}|^2 \tag{2.7}$$

for $p \geqslant 1$. Use Sobolev inequality

$$\|(-v)^{p/2}\|_{L^{\frac{2n}{n-1}}} \leqslant C_1\|(-v)^{p/2}\|_{H^1}. \tag{2.8}$$

Then, we have

$$\|v\|^p_{L^{\frac{np}{n-1}}} \leqslant C_2 p\|v\|^p_{L^p}. \tag{2.9}$$

Let $\gamma = \frac{n}{n-1}$ and $p = \gamma^j$ where $j = 0, 1, 2 \ldots$. Then by induction on j, we get

$$\begin{aligned}
\|v\|_{L^{\gamma^{j+1}}} &\leqslant C_2^{\frac{1}{\gamma^j}}\gamma^{\frac{j}{\gamma^j}}\|v\|_{L^{\gamma^j}} \\
&\leqslant \ldots\ldots \\
&\leqslant C_2^{\sum_{k=0}^{j}\frac{1}{\gamma^k}}\gamma^{\sum_{k=1}^{j}\frac{k}{\gamma^k}}\|v\|_{L^1} \\
&\leqslant C_2^{\sum_{k=0}^{\infty}\frac{1}{\gamma^k}}\gamma^{\sum_{k=1}^{\infty}\frac{k}{\gamma^k}}(C_3+1)Vol(M) \\
&= C_4.
\end{aligned} \tag{2.10}$$

Let $j \to \infty$, we have

$$\|v\|_{L^\infty} \leqslant C_4. \tag{2.11}$$

C^1-estimate: By interpolation, we have the following estimate. We refer to [33] as an introductory textbook for those who are not familiar with this in PDE.

$$\|\nabla\varphi\|_{L^\infty} \leqslant C\big(\|\varphi\|_{L^\infty} + \|\Delta\varphi\|_{L^\infty}\big). \tag{2.12}$$

C^2-estimate: Yau [114] proved the following result in his paper

$$\begin{aligned}
\Delta_\varphi[e^{-C\varphi}(m+\Delta\varphi)] \geqslant{}& e^{-C\varphi}[\Delta F - m^2\inf_{i\neq j}(R_{i\bar{i}j\bar{j}}) - Cm(m+\Delta\varphi)] \\
&+ e^{-C\varphi}[C + \inf_{i\neq j}(R_{i\bar{i}j\bar{j}})]\ e^{F/(1-m)}(m+\Delta\varphi)^{m/(m-1)}.
\end{aligned} \tag{2.13}$$

Choose C large enough such that $C + \inf_{i\neq j}(R_{i\bar{i}j\bar{j}}) > 1$ and we can assume that $e^{-C\varphi}(m+\Delta\varphi)$ attains its maximum at a point p. Then $\Delta_\varphi[e^{-C\varphi}(m+\Delta\varphi)] \leqslant 0$ at p. Consider the right-hand side of the equation, $(m + \Delta\varphi)(p)$ has an upper bound depending only on M and F. Also since $m + \Delta\varphi = g^{i\bar{j}}\widetilde{g}_{i\bar{j}} > 0$, we get the following inequality

$$0 < m + \Delta\varphi \leqslant C_5 e^{C_6(\varphi - \inf\varphi)}. \tag{2.14}$$

Thus, we get C^2-estimate.

$C^{2,\alpha}$-estimate: We use the theory of Evans-Krylov estimate [45]. Let $F : \mathbb{R}^{n^2} \to \mathbb{R}$ be a smooth convex elliptic function, u be a smooth solution of

$$F(D^2u) = 0 \tag{2.15}$$

on a bounded domain Ω. Then there exist a constant $C\big(\Omega', \|D^2u\|_{L^\infty}, F\big)$ such that

$$\|D^2u\|_{C^\alpha(\Omega')} \leqslant C\big(\Omega', \|D^2u\|_{L^\infty}, F\big), \tag{2.16}$$

where $\Omega' \subset\subset \Omega$.

Now we are ready to complete the proof. Differential the equation in S we have

$$\det(g_{i\bar{j}} + \varphi_{q,i\bar{j}})g_q^{i\bar{j}}\left(\frac{\partial^2}{\partial_i\partial_{\bar{j}}}\frac{\partial\varphi_q}{\partial z^p} + \partial_p g_{i\bar{j}}\right) = Vol(M)\frac{\partial_p(e^{t_qF}\det g_{i\bar{j}})}{\int_M e^{tqF}}. \tag{2.17}$$

The operator on the left-hand side of equation is uniformly elliptic and the sequence of coefficients are bounded in the norm $|\bullet|_{0,\alpha}$ by the estimate above. The Schauder estimate shows that $\left|\frac{\partial\varphi_q}{\partial z^p}\right|_{2,\alpha}$ are also bounded. From this, we know that $|\varphi_q|_{3,\alpha}$ is bounded, and then the coefficients are bounded in the norm $|\varphi_{qi\bar{j}}|_{1,\alpha}$. The Schauder estimate again yields that $\left|\frac{\partial\varphi_q}{\partial z^p}\right|_{3,\alpha}$ is bounded. We go on this way

$$\cdots \Rightarrow \left|\frac{\partial\varphi_q}{\partial z^i\partial\bar{z}^j}\right|_{k,\alpha} \Rightarrow \left|\frac{\partial\varphi_q}{\partial z^p}\right|_{k+2,\alpha} \Rightarrow \left|\varphi_q\right|_{k+3,\alpha} \Rightarrow \left|\frac{\partial\varphi_q}{\partial z^i\partial\bar{z}^j}\right|_{k+1,\alpha} \Rightarrow \cdots$$

We then have that $\forall k$, the sequence $\{\varphi_q\}_{q=1}^\infty$ is bounded in the norm $|\bullet|_{k,\alpha}$. By Ascoli's theorem, the sequence has a pointwise convergent subsequence and the limit lies in $C^k(M)$. Since k is arbitrary, we get a smooth solution φ_T of $\frac{\det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{\det g_{i\bar{j}}} = Vol(M)\frac{e^{tF}}{\int_M e^{tF}}$ where $t = T$. This proves S is closed.

Proof of Calabi conjecture by Ricci flow method [17]: Let M be a Kähler manifold with metric g and $T = \frac{\sqrt{-1}}{2\pi}T_{i\bar{j}}dz^i \wedge d\bar{z}^j$ represents the first Chern class $c_1(M)$. We consider the complex version of Hamilton equation of the following type

$$\begin{cases} \dfrac{\partial\widetilde{g}_{i\bar{j}}}{\partial t} = T_{i\bar{j}} - \widetilde{R}_{i\bar{j}}, \\ \widetilde{g}_{i\bar{j}}(0) = g_{i\bar{j}}, \end{cases} \tag{2.18}$$

where $\widetilde{R}_{i\bar{j}}$ denotes the Ricci curvature of $\widetilde{g}$. If we can prove the solution of the equation above exists for all time t and $\widetilde{g}_{i\bar{j}}(t)$ converges to the limit metric $\widetilde{g}_{i\bar{j}}(\infty)$ as $t \to \infty$, then we will get the metric that we want. Bythe global $\partial\bar{\partial}$-lemma, there exists $f \in C^\infty(M, \mathbb{R})$ such that

$$T_{i\bar{j}} - R_{i\bar{j}} = f_{i\bar{j}}. \tag{2.19}$$

Also, we have a smooth real-valued function $u \in M \times [0, t_0)$, $0 < t_0 \leqslant \infty$, with $u(0) = 0$ such that

$$\widetilde{g}_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}. \tag{2.20}$$

Then we can equivalently write

$$\frac{\partial u}{\partial t} = \log\det(g_{i\bar{j}} + u_{i\bar{j}}) - \log\det(g_{i\bar{j}}) + f + \varphi(t), \tag{2.21}$$

in which φ should satisfy the compatibility condition

$$\int_M e^{\frac{\partial u}{\partial t} - f} = e^{\varphi(t)}\ Vol(M). \tag{2.22}$$

Now we reduce the problem to the following nonlinear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \log\det(g_{i\bar{j}} + u_{i\bar{j}}) - \log\det(g_{i\bar{j}}) + f, \\ u|_{t=0} = 0, \end{cases} \tag{2.23}$$

where u is a solution in the maximal time interval $[0, t_0)$, and $\widetilde{g}$ defines a Kähler metric on M for $\forall\ t \in [0, t_0)$. Define a normalization $\widetilde{u}$ to be

$$\widetilde{u} = u - \frac{1}{Vol(M)} \int_M u, \tag{2.24}$$

then $t_0 = \infty$ and $\widetilde{u}$ is uniformly bounded. This makes no difference with continuous method.

The main result is that $\widetilde{u}(t)$ converges in C^∞ topology to a smooth function $\widetilde{u}(\infty)$ as $t \to \infty$, and $\frac{\partial u}{\partial t}$ converges to a constant in C^∞ topology. From this, we conclude that $\widetilde{R}_{i\bar{j}}(\infty) = T_{i\bar{j}}$. This proves the problem. We know that $\frac{\partial u}{\partial t}$ satisfies the equation

$$\begin{cases} (\widetilde{\Delta} - \dfrac{\partial}{\partial t})\dfrac{\partial u}{\partial t} = 0, \\ \dfrac{\partial u}{\partial t}(x, 0) = f(x), \end{cases} \tag{2.25}$$

By the maximum principle for the parabolic equation that for $0 < t_1 < t_2$

$$\begin{aligned} \sup_{x\in M} \frac{\partial u}{\partial t}(x, t_2) &< \sup_{x\in M} \frac{\partial u}{\partial t}(x, t_1) < \sup_{x\in M} f(x), \\ \inf_{x\in M} \frac{\partial u}{\partial t}(x, t_2) &> \inf_{x\in M} \frac{\partial u}{\partial t}(x, t_1) > \inf_{x\in M} f(x). \end{aligned} \tag{2.26}$$

We define

$$\begin{aligned} \varphi_n(x, t) &= \sup_{x\in M} \frac{\partial u}{\partial t}(x, n-1) - \frac{\partial u}{\partial t}(x, n-1+t), \\ \psi_n(x, t) &= \frac{\partial u}{\partial t}(x, n-1+t) - \inf_{x\in M} \frac{\partial u}{\partial t}(x, n-1), \\ \omega(t) &= \sup_{x\in M} \frac{\partial u}{\partial t}(x, t) - \inf_{x\in M} \frac{\partial u}{\partial t}(x, t). \end{aligned} \tag{2.27}$$

We need to give an estimate of $\omega(t)$.

Oscillation Decay: The following Harnack inequality of parabolic equation on compact

Riemannian manifold is just a modification of Li-Yau's work. See more details in [73].

Theorem 2.1 (Harnack inequality)**.** *Let M be a compact Riemannian manifold with dimension n. $g_{ij}(t)$ de a family of Riemannian metric on M such that $\exists\ C_1, C_2, C_3, K > 0$*

$$C_1 g_{ij}(0) \leqslant g_{ij}(t) \leqslant C_2 g_{ij}(0), \tag{2.28}$$

$$\left|\frac{\partial g_{ij}}{\partial t}\right| \leqslant C_3 g_{ij}(0), \tag{2.29}$$

$$R_{ij}(t) \geqslant -K g_{ij}(0). \tag{2.30}$$

If $\varphi(x,t)$ is a positive solution for the equation

$$(\Delta_t - \frac{\partial}{\partial t})\varphi = 0 \tag{2.31}$$

on $M \times [0,\infty)$, then for $\forall\alpha > 0$ and $0 < t_1 < t_2 < \infty$ we have

$$\sup_{x\in M} \varphi(x,t_1) \leqslant \inf_{x\in M} \varphi(x,t_2)(\frac{t_2}{t_1})^{n/2} e^{\frac{C_2^2 diam(M,g_0)^2}{4(t_2-t_1)} + \left[\frac{n\alpha K}{2(\alpha-1)} + C_2C_3\left(n + \sup\|\nabla^2 \log\varphi\|\right)\right](t_2-t_1)}. \tag{2.32}$$

It is easy to see $\widetilde{g}_{i\bar{j}}$ satisfies (2.30) , also φ_n and ψ_n in (2.27) satisfy (2.31) and are positive. Using Harnack inequality, we get

$$\begin{aligned}
&\sup_{x\in M}\frac{\partial u}{\partial t}(x,n-1) - \inf_{x\in M}\frac{\partial u}{\partial t}(x,n-\frac{1}{2}) \leqslant \gamma\left[\sup_{x\in M}\frac{\partial u}{\partial t}(x,n-1) - \sup_{x\in M}\frac{\partial u}{\partial t}(x,n)\right],\\
&\sup_{x\in M}\frac{\partial u}{\partial t}(x,n-\frac{1}{2}) - \inf_{x\in M}\frac{\partial u}{\partial t}(x,n-1) \leqslant \gamma\left[\inf_{x\in M}\frac{\partial u}{\partial t}(x,n) - \inf_{x\in M}\frac{\partial u}{\partial t}(x,n-1)\right],
\end{aligned} \tag{2.33}$$

hence we have

$$\omega(n) \leqslant \left(\frac{\gamma-1}{\gamma}\right)\omega(n-1). \tag{2.34}$$

Denote $a = \log(\frac{\gamma}{\gamma-1})$ and by the fact $\omega(t)$ is monotone decreasing, then we can choose a constant $C_4 > 0$ such that

$$\omega(t) \leqslant C_4 e^{-at}. \tag{2.35}$$

Energy Decay: Let us define

$$\begin{aligned}
\varphi(x,t) &= \frac{\partial u}{\partial t} - \frac{1}{Vol(M)}\int_M \frac{\partial u}{\partial t} dV_{\widetilde{g}},\\
E &= \frac{1}{2}\int_M \varphi^2 dV_{\widetilde{g}}.
\end{aligned} \tag{2.36}$$

Let t be large enough such that $\sup_{x\in M}\varphi(x,t) < \omega(t) < \frac{1}{2}$ We can directly compute

$$\begin{aligned}
\frac{d}{dt}E &= \int_M \varphi\frac{\partial\varphi}{\partial t} dV_{\widetilde{g}} + \frac{1}{2}\int_M \varphi^2\frac{\partial}{\partial t}(dV_{\widetilde{g}})\\
&= \int_M\left(\frac{\partial u}{\partial t} - \frac{1}{Vol(M)}\int_M\frac{\partial u}{\partial t}dV_{\widetilde{g}}\right)\left[\frac{\partial^2 u}{\partial t^2} - \frac{1}{Vol(M)}\int_M\frac{\partial u}{\partial t}\frac{\partial}{\partial t}\log\det(g_{i\bar{j}} + u_{;i\bar{j}})dV_{\widetilde{g}}\right.\\
&\qquad\left. - \frac{1}{Vol(M)}\int_M\frac{\partial^2 u}{\partial t^2}dV_{\widetilde{g}}\right]dV_{\widetilde{g}} + \frac{1}{2}\int_M\varphi^2\frac{\partial}{\partial t}\log\det(g_{i\bar{j}} + u_{;i\bar{j}})dV_{\widetilde{g}}\\
&= \int_M\left(\frac{\partial u}{\partial t} - \frac{1}{Vol(M)}\int_M\frac{\partial u}{\partial t}dV_{\widetilde{g}}\right)\left[\widetilde{\Delta}\left(\frac{\partial u}{\partial t}\right) - \frac{1}{Vol(M)}\int_M\frac{\partial u}{\partial t}\widetilde{\Delta}\left(\frac{\partial u}{\partial t}\right)dV_{\widetilde{g}}\right]dV_{\widetilde{g}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_M \varphi^2\widetilde{\Delta}\left(\frac{\partial u}{\partial t}\right) dV_{\widetilde{g}}\\
=&\int_M \frac{\partial u}{\partial t}\widetilde{\Delta}\left(\frac{\partial u}{\partial t}\right) dV_{\widetilde{g}}+\frac{1}{2}\int_M \varphi^2\widetilde{\Delta}\left(\frac{\partial u}{\partial t}\right) dV_{\widetilde{g}}\\
=&\int_M (-1-\varphi)\left|\widetilde{\nabla}\frac{\partial u}{\partial t}\right|^2 dV_{\widetilde{g}}\\
\leqslant&-\frac{1}{2}\int_M |\widetilde{\nabla}\varphi|^2 dV_{\widetilde{g}}\\
\leqslant&-\frac{1}{2}\lambda_1(t)\int_M \varphi^2 dV_{\widetilde{g}}.
\end{aligned}\tag{2.37}
$$

This implies that we can choose $C_5 > 0$ such that

$$\frac{d}{dt}E \leqslant -C_5 E. \tag{2.38}$$

Since $dV_{\widetilde{g}}$ is uniformly equivalent to dV_g, there exists $C_6 > 0$

$$\int_M \varphi^2 dV_g \leqslant C_6 e^{-C_5 t}. \tag{2.39}$$

Now we are in the position to prove the main result. For any $0 < s < s'$, we have

$$
\begin{aligned}
\int_M |\widetilde{u}(x,s')-\widetilde{u}(x,s)|\ dV_g \leqslant&\int_M\int_s^{s'}\left|\frac{\partial\widetilde{u}}{\partial t}(x,t)\right|\ dt\ dV_g\\
=&\int_s^{s'}\int_M\left|\frac{\partial u}{\partial t}-\frac{1}{Vol(M)}\int_M\frac{\partial u}{\partial t}\right|\ dV_g\ dt\\
\leqslant&\int_s^{s'}\int_M|\varphi|\ dV_g\ dt\\
&+\int_s^{s'}\int_M\frac{1}{Vol(M)}\left|\int_M\frac{\partial u}{\partial t}\ dV_{\widetilde{g}}-\int_M\frac{\partial u}{\partial t}\ dV_g\right|\ dV_g\ dt\\
\leqslant&\int_s^{\infty}\sqrt{Vol(M)}\sqrt{\int_M\varphi^2 dV_g}\ dt+Vol(M)\int_s^{\infty}\omega(t)\ dt\\
\leqslant&\ C_7\int_s^{\infty}e^{-C_5t/2}dt+C_8\int_s^{\infty}e^{-at}dt.
\end{aligned}\tag{2.40}
$$

The computation above shows that $\widetilde{u}(x,t)$ converge in L^1 norm. If $\widetilde{u}$ does not converge in C^∞ topology, then there exist $\epsilon > 0$ some integer $r > 0$ and a sequence $\{t_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} t_i = +\infty$

$$\|\widetilde{u}(t_n)-\widetilde{u}(\infty)\|_{c^r} \geqslant \epsilon \tag{2.41}$$

for any n. But $\widetilde{u}(t_n)$ are bounded in C^∞ topology, so there exists a subsequence that converge to $\widetilde{u}(\infty)$ in C^∞ topology. It is a contradiction.

3 KE metric with $c_1(M) \leqslant 0$

We know that a Kähler manifold is Kähler-Einstein if and only if $\rho = \lambda\omega$. Moreover, we can reduce it to the cases $\lambda = 0, 1, -1$. The conjecture is to ask if there is a Kähler form such that $\rho = \lambda\omega$ when $\lambda c_1(M)$ is a Kähler class.

When $\lambda = 0$. It is just a corollary of Calabi conjecture. We now discuss the case $\lambda = -1$. In this case, we should consider the equation

$$\frac{\det\left(g_{i\bar{j}} + \partial_{i\bar{j}}\varphi\right)}{\det g_{i\bar{j}}} = e^{F+\varphi}. \tag{3.1}$$

The equation have a unique solution such that $g_{i\bar{j}} + \partial_{i\bar{j}}\varphi$ defines a Kähler metric.

Theorem 3.1. *Let (M, ω) be a Kähler manifold, $-c_1(M)$ is a Kähler class, then there is a Kähler form such that $\rho = -\omega$. This means M is Kähler-Einstein.*

Prove Theorem 3.1 by continuous method

To prove Theorem 3.1, we need the following result about a Monge-Ampère equation.

Theorem 3.2. *Let M be a compact Kähler manifold with Kähler metric g, $\dim M = m$. Let $F(x)$ be any smooth function defined on M. Then for any constant $k > 0$, there exists a unique smooth function φ, up to a constant, such that*

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{f+k\varphi} \det(g_{i\bar{j}}) \tag{3.2}$$

and $\widetilde{g}_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$ defines a Kähler metric.

First of all, let us prove the uniqueness of the solution of (3.2) . If φ and $\widetilde{\varphi}$ are two solutions, then denote $F(x,t) = f + kt$, $\frac{\partial F}{\partial t} \geqslant 0$ and we have

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{F(x,\varphi)} \det(g_{i\bar{j}}) \tag{3.3}$$

and

$$\det(g_{i\bar{j}} + \widetilde{\varphi}_{i\bar{j}}) = e^{F(x,\widetilde{\varphi})} \det(g_{i\bar{j}}). \tag{3.4}$$

Choose a normal local coordinate of g_φ such that $\widetilde{\varphi}_{i\bar{j}} - \varphi_{i\bar{j}} = \delta_{ij}$. By AM-GM mean inequality, we have

$$\begin{aligned} e^{F(x,\widetilde{\varphi})-F(x,\varphi)} &= \frac{\det(g_{i\bar{j}} + \widetilde{\varphi}_{i\bar{j}})}{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})} \\ &= \prod_{i=1}^{m} \left[1 + g_\varphi^{i\bar{i}}(\widetilde{\varphi}_{i\bar{i}} - \varphi_{i\bar{i}})\right] \\ &\leqslant \left[1 + \frac{1}{m}\Delta_\varphi(\widetilde{\varphi} - \varphi)\right]^m. \end{aligned} \tag{3.5}$$

By the mean value theorem

$$\begin{aligned} F(x,\widetilde{\varphi}) - F(x,\varphi) &= \int_{\varphi(x)}^{\widetilde{\varphi}(x)} \frac{\partial F}{\partial t}(x,t)dt \\ &= \frac{\partial F}{\partial t}(x,\bar{t})[\widetilde{\varphi}(x) - \varphi(x)], \end{aligned} \tag{3.6}$$

where $\bar{t}(x) \in \left[\inf\{\varphi(x), \widetilde{\varphi}(x)\}, \sup\{\varphi(x), \widetilde{\varphi}(x)\}\right]$. Assume that $\widetilde{\varphi}(x) - \varphi(x)$ attains its maximum at x_0. If $\widetilde{\varphi}(x_0) - \varphi(x_0) > 0$, then $\Delta_\varphi\big(\widetilde{\varphi}(x) - \varphi(x)\big) \geqslant 0$ in a neighborhood of x_0. By the maximal principle and the compactness of M, $\widetilde{\varphi} - \varphi$ is a constant function. We switch the roles of $\widetilde{\varphi}$ and φ and repeat the discussion above. Finally, we can deduce $\varphi - \widetilde{\varphi}$ is a constant function. This prove the uniqueness.

Now, let us prove the existence. We are going to prove this using continuous method. Consider the set

$$S = \left\{ t \in [0,1] \middle| \text{ The equation } \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{tf+k\varphi} \det(g_{i\bar{j}}) \text{ has a solution in } C^\infty(M) \right\}.$$

Obviously $0 \in S$ for $\varphi = 0$ is a solution. If we can show the set S is both open and closed, this will imply that $1 \in S$.

To see S is open, we consider a map $G : A \to B$. Where

$$\begin{aligned} A &:= \left\{ \varphi \in C^\infty(M) \middle| \sum_{i,j} (g_{i\bar{j}} + \varphi_{i\bar{j}}) dz^i \otimes d\bar{z}^j \text{ defines a metric on } M \right\}, \\ B &:= \left\{ f \in C^\infty(M) \middle| \int_M f = Vol(M) \right\}, \\ G(\varphi) &:= \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) \det(g_{i\bar{j}})^{-1}. \end{aligned}$$

As we have discussed in previous section, that G is an open map. Also $G_1(\varphi) = e^{-k\varphi}$ is an open map. Thus, $G_2(\varphi) = G(\varphi)G_1(\varphi)$ is open map. This proves S is open.

To see S is closed. Let $\{t_q\}_{q=1}^\infty$ is a sequence in S where $\lim_{q\to\infty} t_q = T$ then we have a sequence $\varphi_q \in S$ such that

$$\det(g_{i\bar{j}} + \varphi_{qi\bar{j}}) = e^{t_q f + k\varphi_q} \det(g_{i\bar{j}}). \tag{3.7}$$

Differential the equation we have

$$\det(g_{i\bar{j}} + \varphi_{qi\bar{j}}) g_q'^{i\bar{j}} \left(\frac{\partial^2}{\partial_i \partial_{\bar{j}}} \frac{\partial \varphi_q}{\partial z^p} + \partial_p g_{i\bar{j}} \right) = \partial_p \left(e^{t_q f + k\varphi_q} \det g_{i\bar{j}} \right). \tag{3.8}$$

Let $\varphi_{qi\bar{j}}$ achieves its maximum at x_0, we have

$$e^{t_q f + k\varphi_q} = \frac{\det(g_{i\bar{j}} + \varphi_{qi\bar{j}})}{\det(g_{i\bar{j}})} \leqslant 1. \tag{3.9}$$

This immediately implies $\varphi(x_0) \leqslant -\frac{t_q}{k} F(x_0)$. Similarly we can derive an estimate of $\inf_M \varphi$. Then we get φ_q is bounded in $|\bullet|_{0,\alpha}$ norm. Going through the process in previous section, we can prove φ_q is bounded in $|\bullet|_{0,\alpha}$ norm. This shows that the sequence of coefficients of (3.8) is bounded in the norm $|\bullet|_{0,\alpha}$. The Schauder estimate shows that $\left|\frac{\partial \varphi_q}{\partial z^p}\right|_{2,\alpha}$ that are also bounded. From this information we know that $|\varphi_q|_{3,\alpha}$ is bounded, then the coefficients of (3.8) is bounded in the norm $|\varphi_{qi\bar{j}}|_{1,\alpha}$. The Schauder estimate again provides that $\left|\frac{\partial \varphi_q}{\partial z^p}\right|_{3,\alpha}$ is bounded. We go on this way

$$\cdots \Rightarrow \left|\tfrac{\partial \varphi_q}{\partial z^i \partial \bar{z}^j}\right|_{k,\alpha} \Rightarrow \left|\tfrac{\partial \varphi_q}{\partial z^p}\right|_{k+2,\alpha} \Rightarrow |\varphi_q|_{k+3,\alpha} \Rightarrow \left|\tfrac{\partial \varphi_q}{\partial z^i \partial \bar{z}^j}\right|_{k+1,\alpha} \Rightarrow \cdots$$

We then have that $\forall k$ the sequence $\{\varphi_q\}_{q=1}^\infty$ is bounded in the norm $|\bullet|_{k,\alpha}$. By Ascoli's theorem the sequence has a pointwise convergent subsequence and the limit lies in $C^k(M)$. For the arbitrariness of k, we get a smooth solution φ_T of $\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{tf+k\varphi} \det(g_{i\bar{j}})$ where $t = T$. This prove S is closed.

Now let us prove Theorem 3.1 . The condition shows that there exists a Kähler form ω such that $-\rho_\omega \in c_1(M)$. By local computation and the $\partial\bar{\partial}$-lemma, we find a smooth function f such that

$$\partial\bar{\partial}\log\det(g_{i\bar{j}}) = g_{i\bar{j}}dz^i \wedge d\bar{z}^j + \partial\bar{\partial}f. \tag{3.10}$$

By Theorem 3.2 we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\varphi - f}\det(g_{i\bar{j}}). \tag{3.11}$$

and $\widetilde{g}_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$ defines a Kähler metric. Combining these results, we have

$$-\partial\bar{\partial}\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = -\sqrt{-1}(g_{i\bar{j}} + \varphi_{i\bar{j}})dz^i \wedge d\bar{z}^j. \tag{3.12}$$

Hence we have found the metric that we are looking for.

The metric is unique. Otherwise, let $\bar{g}$ be another such metric. Since $\bar{g}, \widetilde{g} \in [g]$, that is, they are lied in the same de Rham class, we can find a smooth function $\psi \in C^\infty(M)$ such that $\bar{g}_{i\bar{j}} = g_{i\bar{j}} + \psi_{i\bar{j}}$ and ψ also satisfty the equation (3.11)

$$\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = e^{\psi - f}\det(g_{i\bar{j}}) \tag{3.13}$$

and (3.12)

$$-\partial\bar{\partial}\log\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = -\sqrt{-1}(g_{i\bar{j}} + \psi_{i\bar{j}})dz^i \wedge d\bar{z}^j. \tag{3.14}$$

Combining (3.11) (3.12) we have

$$\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = e^{\psi + C - f}\det(g_{i\bar{j}}) \tag{3.15}$$

for some constant C. That means $\psi + C$ is also a solution of (3.11). So we get $\varphi - \psi$ is a constant, $\bar{g} = \widetilde{g}$. The Kähler-Einstein metric is unique.

4 Stability with GIT analysis

As we can see, KE and cscK metrics minimize or maximize some functionals which means they are locally stable. In this section, we will see their relationship with algebraic stability, and we will explain why forming moduli of algebraic varieties should be a GIT problem.

We want to form a moduli space of polarized algebraic varieties [71]. The polarization allows us to embed X into a projective space by Kodaira [54]. In fact, for X smooth, a theorem of Matsusaka tells us that r can be chosen uniformly amongst all (X, L) with the same Hilbert polynomial $\mathcal{P}(r) = \chi(X, L^r)$. Moreover, we can also assume that all higher cohomology groups $H^{\geqslant 1}(X, L^r)$ vanish so that $H^0(X, L^r)$ has dimension $\mathcal{P}(r)$, and that any two (X_i, L_i) are isomorphic if and only if their embeddings differ by a projective linear map. Then (X, L) defines a point in the Hilbert scheme of subvarieties of $\mathbb{CP}^n$, we must classify the choices of isomorphism, i.e. take the GIT quotient of Hilb by $SL(N+1, \mathbb{C})$.

By abstract GIT, any choice of $SL(N+1, \mathbb{C})$-equivariant (anti-)ample line bundle on Hilb gives rise to a notion of stability for (X, L). There are many such examples, and we describe some of those whose associated weights can all be characterized in terms of the weights on the fibre at the point in Hilb. The Hilbert-Mumford criterion requires us to consider $\mathbb{C}^* < SL(N+1, \mathbb{C})$ orbits of $X \subset \mathbb{CP}^n$. This gives rise to a $\mathbb{C}^*$-equivariant flat family, or test configuration, $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$. The weight $w_{r,k}$ of the $\mathbb{C}^*$-action on fibre is

$$w_{r,k} = a_{n+1}(r)k^{n+1} + a_n(r)k^n + \cdots, \tag{4.1}$$

where

$$a_i(r) = a_{i,n}r^n + a_{i,n-1}r^{n-1} + \cdots. \tag{4.2}$$

Hilbert-Mumford criterion requires $w_{r,k} \succ 0$. Donaldson's refinement of Tian's original notion requires one to pick a test configuration first and then choose $r \gg 0$. The coefficient $a_{n+1,n} > 0$ is Donaldson's version [40] of the Futaki invariant of the $\mathbb{C}^*$-action on (X, L).

4.1 Moment map

Fix a metric on $\mathbb{C}^{N+1}$ and so g_{FS} on $\mathbb{CP}^N$ and an induced hermitian metric on $\mathcal{O}(-1)$. This induces the symplectic form ω_{FS} on a smooth $X \subset \mathbb{CP}^N$. This induces a natural symplectic, in fact Kähler, structure on (any smooth subset of smooth points of) Hilb:

$$\Omega(v_1, v_2) := \int_X w_{FS}(v_1, v_2)\frac{\omega_{FS}^n}{n!}, \tag{4.3}$$

where the v_i's are the normal components of holomorphic vector fields along $X \subset \mathbb{CP}^N$. This is also (a multiple of) the first Chern class of a natural line bundle on Hilb coming from the "Deligne pairing" of $\mathcal{O}_X(1)$ with itself $(n+1)$-times [91]. Let $m : \mathbb{CP}^N \to \mathfrak{su}(N+1)^*$ denote the usual moment map. Then Donaldson [39] defined the moment map for $SU(N+1) \curvearrowright (Hilb, \Omega)$ takes $X \subset \mathbb{CP}^N$ to a multiple of its centre of mass in $\mathfrak{su}(N+1)^*$:

$$\mu(X) := \int_X m\frac{w_{FS}^n}{n!}. \tag{4.4}$$

The zeros of the moment map correspond to the balanced varieties $X \subset \mathbb{CP}^N$.

4.2 Asymptotics of Bergman Kernel

Fix a metric on (X, L) (e.g. by picking a metric on $H^0(L)$ and then induce the Fubini-Study metric on $X \subset P(H^0(L^*))$ and $L = \mathcal{O}(1)$. This then induces one on L^r for all r, and so L^2-metrics on $H^0(L^*)$ for all r. Picking an L^2-orthonormal basis $s_i \in H^0(L^*)$, we can then define, for each r, the Bergman kernel:

$$B_r(x_1, x_2) := \Sigma_i s_i(x_1) \otimes s_i(x_2)^* \tag{4.5}$$

on $X \times X$. This is the integral kernel for the L^2-orthogonal projection of C^∞ sections of L^r onto holomorphic sections. Restricting to the diagonal gives $B_r(x) := \Sigma_i |s_i(x)|^2$. So the balanced condition is equivalent to $B_r(x)$ being constant on X.

The importance of Bergman kernel comes from Gang Tian's thesis [95]. His idea is to approximate the Kähler metrics by the projective embedding. Let $\iota_k : X \to \mathbb{CP}^{N_k}$ be the embedding induced by a given basis of $H^0(X, L^k)$, then the projective metrics read as $\omega_k = \frac{1}{k}\iota_k\omega_{FS}$ in the same cohomology class as ω.

Theorem 4.1 (Tian-Ruan [81]). *For any polarized (X, L, ω), we can choose ι_k such that ω_k converges to ω in C^∞ as $k \to \infty$.*

As $r \to \infty$, $B_r(x)$ has an asymptotic expansion [95, 117, 34, 80, 63]

$$B_r(x) \sim r^n + \frac{1}{2\pi} s(x) r^{n-1} + O(r^{n-2}). \tag{4.6}$$

Roughly speaking, the balanced metrics should tend towards cscK metrics with $[w] = [c_1(L)]$. What we have seen so far should motivate the following results.

Theorem 4.2 (Donaldson)**.** *Suppose that Aut(X,L) is discrete and (X, L^k) is balanced for all sufficiently large k. Suppose that the metrics ω_k converge in C^∞ to some limit ω_∞ as $k \to \infty$. Then ω_∞ has constant scalar curvature. The converse is also true. Suppose that ω_∞ is a Kähler metric in the class $2\pi c_1(L)$ with constant scalar curvature. Then (X, L^k) is balanced for large enough k and the sequence of metrics ω_k converge in C^∞ to the limit ω_∞ as $k \to \infty$.*

We hope that the existence of a constant scalar curvature metric should be related to some appropriate algebraic geometric notion of stability. The theorems stated above show that this question can be reduced to, on the one hand, the finite-dimensional issue of the relation between the balanced condition and stability and, on the other hand, to the question of the convergence of the metrics ω_k as $k \to \infty$. In principle, one might be able to prove the existence of the constant scalar curvature metrics by directly showing that the ω_k converge without using PDE theory, but it is hard to see how one might achieve this. Even in the classical case of Riemann surfaces it is hard to see how one could obtain this convergence without the existence of the constant curvature metric.

This result was due to Donaldson [39]. Tian had previously proved K-semistability for KE metrics [99], and a related convergence result for the sequences of Fubini-Study metrics [95], following a suggestion of Yau [115]. Using [39], Mabuchi proved that cscK manifolds with automorphisms are Chow polystable if the automorphisms satisfy a certain stability condition [66]. Donaldson [41] then showed that cscK $\Longrightarrow$ K-semistable without any condition on the automorphisms. The uniqueness was originally proved by Bando-Mabuchi [5] for KE metrics, by Chen [23] for cscK metrics when $c_1 \leqslant 0$, then by Donaldson for the case of general cscK with finite automorphisms. Again, the finite automorphisms condition was relaxed by Mabuchi and by Chen-Tian [103] in the more general setting of extremal metrics and Kähler non-projective metrics.

4.3 Moduli of bundle

Before we work on the stability of variety, let us see a very similar theory which is almost completely worked out. Just as most mathematicians have done, when we start to develop a new theory, we always seek for the motivations from other similar theories which have been well-developed. We have the following picture that illustrate how we can do GIT analysis on the moduli spaces of bundles over a polarized variety.

$$
\begin{tikzcd}
\text{Stability of bundle} \arrow[r, shift left, "\text{Kempf-Ness}"] \arrow[d, leftrightarrow, "\text{MGMS}"', "\text{Hilbert-Mumford}"] \arrow[dr, leftrightarrow] & \text{Bananced } X \to Gr(N(r)) \arrow[l, shift left, "\text{Wang}"] \arrow[d, shift left, "r\to\infty"] \\
\text{Slope stable} \arrow[r, leftrightarrow, "\text{Donaldson-Uhlenbeck-Yau}"] & \text{HMY} \arrow[u, shift left, "\text{Donaldson}"]
\end{tikzcd}
$$

The formal infinite dimensional gauge theory was described by Atiyah-Bott [2]. Fixing a compatible hermitian metric on L and inducing a Kähler form ω on X, then we find $\mathcal{A} := \{\nabla \text{ is a linear connection } | \nabla(g) = 0,\ F_A^{0,2} = 0\}$ inherits a natural Kähler structure with the symplectic form given by $\Omega(a, b) := \int_X tr(a \wedge b) \wedge \omega^{n-1}$ for $a, b \in \Omega^1(End\ E)$ the tangent vectors to $\mathcal{A}$. The action $U(E) \curvearrowright \mathcal{A}$ has a moment map

$$m(A) = F_A^{1,1} \wedge \omega^{n-1} - \lambda Id\omega^n \in \Omega^{2n}(\mathfrak{su}(E)), \tag{4.7}$$

where $\lambda = 2\pi i\mu(E)/\int_X \omega^n$ is a topological invariant and $\mu(E)$ is the slope of E. The zeros of the moment map are Hermitian-Yang-Mills connections. An infinite dimensional version of Kempf-Ness theorem would be formulated in this way: In a polystable orbit of $GL(E)$, there should be a HYM connection (i.e. a metric whose associated Chern connection is HYM; we call this a HYM metric), unique up to the action of U(E), as conjectured by Hitchin and Kobayashi.

We want to form an algebraic moduli space of bundles E over (X, L) of fixed topological type. More generally, to get a compact moduli space, we have to consider the coherent sheaves E of the same Hilbert polynomial $\chi(E(r))$ where those $E(r) := E \otimes L^r$ have no higher cohomology and are generated by their holomorphic sections for $r \gg 0$. We use its monic version, the reduced Hilbert polynomial

$$p_E(r) := r^n + \frac{a_1}{a_0} r^{n-1} + \cdots, \tag{4.8}$$

then E is stable if and only if $p_F(r) \prec p_E(r)$ holds for all coherent subsheaves $F \to E$ in the following sense (depending on the line bundle chosen on the Quot scheme):

- Gieseker stable: $p_F(r) < p_E(r)$ for all $r \gg 0$.

- Slope stable: $\mu(F) < \mu(E)$.

The Gieseker stability and the slope stability coincide on curves X. The slope stability corresponds to taking a certain semi-ample line bundle on the Quot scheme. Roughly speaking, it is given by restricting sheaves to high degree complete intersection curves in X and using the usual line bundle for moduli of bundles on the curve. GIT needs modifications in this situation; and so far, this has been carried out only for surface [55].

As in the varieties case, we can also talk about the balance. Fix compatible hermitian metrics on L and E induces a Kähler form on X and an L^2-metric on $\mathbb{CP}^N \cong H^0(E(r))$. Then there are actions of $SU(N) < SL(N, \mathbb{C})$ on Gr (Grassmannian of quotients of $\mathbb{C}^N$), inducing a moment map $m : Gr \to \mathfrak{su}(N)^*$ and an action $SU(N) \curvearrowright Maps(X, Gr)$. Its moment map is the integral of (the pullback of) m over X, so we can again talk about the balance and the asymptotics as $r,\ N \to \infty$.

In the proof of Donaldson's conjectures [37], Wang [108] shows that the existence of a balanced map is equivalent to the Gieseker polystability of E. The slope stable bundles (which are therefore Gieseker stable) admit balanced maps $X \to Gr$ for $r \gg 0$ [109], and pulling back the canonical quotient connection on Gr and taking limit $r \to \infty$ gives a conformally Hermitian-Yang-Mills connection on E (which is HYM after rescaling). Unfortunately, this is not the way that the results were proved; Wang used the Donaldson-Uhlenbeck-Yau theorem to give an a priori HYM connection which can be compared to the sequence of balanced metrics.

Moduli of variety. There is a very similar story of variety below

$$\begin{tikzcd}
\text{Stability of varieties} \arrow[rr, shift left, "\text{Kempf-Ness}"] \arrow[d, shift left, "\text{Ross-Thomas}"] \arrow[drr, dashed, shift left] & & \text{Bananced } X \in Hilb \arrow[ll, shift left, "\text{Zhang}"] \arrow[d, dashed, shift left, "r\to\infty"] \\
\text{Slope criterion} \arrow[u, dashed, shift left, "\text{Hilbert-Mumford}"] \arrow[rr, dashed, leftrightarrow, "???"] & & \text{cscK} \arrow[u, shift left, "\text{Donaldson}"] \arrow[ull, shift left, "\text{Donaldson-Tian}"]
\end{tikzcd}$$

Instead of letting the dimension N of our quotient problem go to infinity, Donaldson [36] gave a purely infinite dimensional formal symplectic quotient formulation. The group of the Hamiltonian diffeomorphisms acts on X and thus on the space of complex structures which make (X,ω) Kähler: $Ham(X,\omega) \curvearrowright \mathcal{J} := \{\omega\text{-compatible complex structures on } X\}$.

The Kähler structure on X induces one symplectic structure on $\mathcal{J}$ by integration. This is preserved by $Ham(X,\omega)$, and we can ask for a moment map. Considering $C_0^\infty(X)$ (the functions of integral zero) that lies in the dual of the Lie algebra $C^\infty(X,\mathbb{R})/\mathbb{R}$ by integration over ω^n, and setting S_0 to be the topological constant, average of scalar curvature, Fujiki [48] and Donaldson [36] showed that moment map is $S - S_0$. Thus zeros of the moment map correspond to cscK metrics.

We know that differential of a log-norm function is moment map, we want to set up an infinite dimensional analogue. Considering an orbit ih, i.e., in the family of Kähler form $\omega_t = \omega + 2it\partial\bar{\partial}h$, we seek for a function such that $\frac{dM}{dt} = m_h$. By integration on time

$$M(\omega_s) = \int_0^s \int_X (S_t - S_0) h \frac{\omega_t^n}{n!} dt. \tag{4.9}$$

This is precisely the Mabuchi functional or the K-energy [65], defined up to a constant (equivalent to the ambiguity in the choice of a lift of a point to the line bundle above it). Indeed, it can be written as the log-norm functional for a Quillen metric on a line bundle over the space of Kähler metrics [70]. Its critical points are cscK metrics, and one expects that such a metric exists on (X,J) if and only if M is proper on the space of Kähler metrics on (X,J), which is the infinite dimensional analogue of quotient of group action. This actually is the original definition of the Futaki invariant [49] for a smooth polarized manifold (X,L)

with a $\mathbb{C}^*$-action. Noting as above that it is the weight of the induced action on a line led Donaldson to give the more general definition $a_{n+1,n}$ described earlier, for an arbitrary polarized scheme (X, L).

As Donaldson explains in [38], the finite dimensional problem of balanced metrics can be regarded as the quantization of the infinite dimensional problem of cscK metrics, which emerges as the classical limit as $r, N \to \infty$. He proved a quantitative Kempf-Ness theorem that $SU(N+1)$ really approximates $Ham(X, \omega)$ in the sense that its finite dimensional moment map converges to the infinite dimensional one. The symplectic structures and the natural norm functionals and weights tend to their infinite dimensional analogues (the Mabuchi functional and Futaki invariant) as $r \to \infty$ [38]. Also the space of "algebraic metrics" (the restrictions of the Fubini-Study metrics $SL(N+1)\omega_{FS}$ from $\mathbb{CP}^N$) becomes dense in the space of all Kähler metrics as $r, N \to \infty$ [95]. Thus the quantum picture tends to the classical one as $r \to \infty$.

By the analogue of the Kempf-Ness theorem in finite dimensions and taking the infinite-dimensional limit of $r \to \infty$, it is natural to conjecture a Hitchin-Kobayashi correspondence, that is, a variety should admit a cscK metric if and only if it is polystable in a certain sense. In fact, Yau [116] first suggested that there should be a relationship between stability and the existence of KE metrics. Tian [96] proved this for surfaces, and introduced his notion of K-stability, and [99] showed it satisfied by Kähler-Einstein manifolds based on his work with Ding [35]. The definition of K- stability was generalised to more singular test configurations by Donaldson [40] who also showed that cscK implies K-semistability [39]. So it was thought that K-polystability, as defined above, should be the right notion to be equivalent to cscK. Recent explicit examples [1] in the extremal metrics case where there is a similar conjecture due to Székelyhidi [89] suggest that this should be strengthened to the analytic K-polystability, which allows more general analytic test configurations instead of algebraic ones only. In particular, one should allow the line bundle L over the test configuration to be an $\mathbb{R}$-line bundle: an $\mathbb{R}$-linear combination by tensor product of $\mathbb{C}^*$-linearised line bundles. So the most likely Yau-Tian-Donaldson conjecture as things stand at the end of 2005 is the following.

Conjecture 4.1 (Yau-Tian-Donaldson)**.** *(X, L) is analytically K-polystable $\iff$ (X, L) admits a cscK metric. This is unique up to the holomorphic automorphisms of (X, L).*

This would be the right higher dimensional generalisation of the uniformisation theorem for Riemann surfaces. There is very little progress on this conjecture in the $\Longrightarrow$ direction except for projective bundles [13, 53, 79] and Donaldson's deep work on toric surfaces [40]. In the KE case, there are sufficient conditions for the existence given by Tian's α-invariant [93] and Nadel's multiplier ideal sheaf, but no one successfully made these related to stability. Part of the problem, quite apart from the analytical difficulties, is that we do not have a good intrinsic understanding of stability for varieties, i.e., no one has successfully analysed the Hilbert-Mumford criterion for varieties.

5 YTD Conjecture on Fano Manifold

The most ideal theory we might expect would be that every compact manifold with $c_1(M) = \lambda[\omega]$ has a unique KE metric. But an example of Matsushima [67] showed that

$c_1(M) = [\omega]$ could not derive $\rho = \omega$. We wish to add some conditions on M such that the conjecture becomes true. We have discussed the motivation of the KE and cscK metrics. In this section, we outline the ideas and the techniques in the proof of the existence of KE and cscK metric.

Let (M, ω_0) be a Kähler manifold with $c_1(M) = [\omega_0]$, then as we have discussed in Section 1 that the conjecture of Calabi can be reduced to solving the following complex Monge-Ampère equation

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}})e^{F-t\varphi}, \\ \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \end{cases} \tag{5.1}$$

where $F \in C^\infty(M)$ is a given function.

By a slight modification of Yau's proof, we reduced the equation to the C^0-estimate of solutions. Unfortunately, it doesn't always exist. Tian finally solved the Fano cases in dimension 2 [96]. A general question is: which Fano manifolds admit Kähler-Einstein metrics? The idea that the appropriate condition should be in terms of "algebro-geometric stability" was proposed by Yau about 20 years ago, partly by the analogue to the "Kobayashi-Hitchin correspondence" in the case of holomorphic bundles. Over the years, various different notions of stability have been discussed in literatures, both in the Kähler-Einstein case and in the more general situation of constant scalar curvature Kähler metrics on polarised manifolds.

- Futaki: Futaki invariant, 1983.
- Bando, Mabuchi: Mabuchi functional, 1985.
- Tian: α-invariant, 1987.
- Tian: $Lie(Aut(M))$ is reductive, 1990.
- Yau: Test-configuration, algebraic stability, 1990
- Ding: Ding functional, 1992.
- Tian: K-stability, 1997.
- Donaldson: K-stability (formally), 2002.
- Paul: Stable pair, 2012.

The condition proposed by Tian in 1997 first gave the equivalence. Donaldson extended the definition of K-stability introduced by Tian. This extended definition has two good properties: (i) It is purely algebraic; (ii) It does not require the smoothness (or normality) of the limit cycle. Sean Paul's Stable pair [74] also (iii) completely captured the behavior of the Mabuchi energy along the degeneration. In the case of a smooth limit cycle, his definition of the generalized Futaki invariant agrees with the original definition of Ding and Tian[35]. The generalized Futaki invariant (and the corresponding notion of stability) proposed by Donaldson in 2002 [40] only satisfies (iii) in the special case of reduced limit cycle.

5.1 Gromov-Hausdorff limit with non-collapsing condition

We suppose the metric satisfies fixed upper and lower bounds on the Ricci tensor $-C_1 g \leqslant Ric \leqslant C_2 g$. For $V, c > 0$, let $K(n, c, V)$ denote the class of all such polarized Kähler manifold such that the volume of X is V and the "non-collapsing" condition $VolB_r \geqslant c\frac{\pi^n}{n!}r^{2n}$ holds. The connection induces a holomorphic structure on L and for each positive integer k there is a natural L^2 hermitian metric on the space $H^0(X, L^k)$. Consider the minimum of

Bergman function

$$\underline{\varrho}(k,K) := \min_{x\in X} B(x). \tag{5.2}$$

Kodaira embedding theorem asserts that for each fixed X we have $\underline{\varrho}(k,K) > 0$ for large enough k. A famous result [44] can be regarded as an extension of this well-known statement which is both uniform over $K(n,c,V)$ and gives a definite lower bound.

Theorem 5.1. *Given n, c and V, there is an integer k_0 and $b > 0$ such that $\underline{\varrho}(k,K) \geqslant b^2$ for all $X \in K(n,c,V)$.*

The proof involves a combination of the Gromov–Hausdorff convergence theory—developed by Anderson, Cheeger, Colding, Gromoll, Gromov, Tian and others over the past thirty years or so, and the "Hörmander technique" for constructing holomorphic sections. When $n = 2$ the theorem was essentially proved by Tian in [96] and the overall proof is similar. We remark that the original conjecture of Tian in [97] is stated for Kähler metrics on Fano manifolds with a uniform positive lower bound on the Ricci curvature, and this amounts to removing the hypothesis on the upper bound of Ricci curvature in the above theorem. This remains an interesting open question to study in the future.

The above theorem provides the foundations for a bridge between the differential geometric convergence theory and algebraic geometry, which leads to the following result (as indicated by Tian).

Theorem 5.2. *Given n, c and V, there is a fixed k_1 and an integer N such that any X in $K(n,c,V)$ can be embedded in a linear subspace of $\mathbb{CP}^N$ by sections of L^{k_1}. Moreove, if X_j be a sequence in $K(n,c,V)$ with Gromov–Hausdorff limit X_∞. Then X_∞ is homeomorphic to a normal projective variety W in $\mathbb{CP}^N$. After passing to a subsequence and taking a suitable sequence of projective transformations, we can suppose that the projective varieties $X_j \in \mathbb{CP}^N$ converge as algebraic varieties to W.*

Many of the ideas and arguments required to derive this are similar to those of Ding and Tian in [35] that considered Fano manifolds with Kähler–Einstein metrics. Then the limit is a "$\mathbb{Q}$-Fano" variety, as Ding and Tian conjectured.

5.2 Cheeger-Colding theory

The study of the structure of spaces Y, which are pointed Gromov-Hausdorff limits of sequences, $\{(M_i^n, p_i)\}$, of complete, connected Riemannian manifolds whose Ricci curvatures have a definite lower bound systematically presented by Jeff Cheeger and Tobias H. Colding [20, 21, 22]. Most of results are applications of the "almost rigidity" theorems for manifolds of almost nonnegative Ricci curvature [18, 19]. The techniques is the use of the generalized splitting theorem, "volume cone implies metric cone" and (implicitly) integral Toponogov theorems together with the tangent cone analysis of the sort employed in geometric measure theory. The continuity of the volume (of balls) under Gromov-Hausdorff limits also plays a direct role in the discussion. The continuity of the volume in the case was conjectured by Anderson-Cheeger and proved in [32].

5.3 Conical KE metric

Now, we look at a pair (X, D) where D is a smooth divisor in a Kähler manifold X, and study the existence of Kähler-Einstein metrics on X with cone singularities along D. This problem was classically studied on the Riemann surfaces [64, 68, 105], and was first considered in higher dimensions by Tian in [97]. Recently, there is a reviving interest on this generalized problem, mainly due to Donaldson's program [43] on constructing smooth Kähler-Einstein metrics on X by varying the angle along an anti-canonical divisor. There are many subsequent works, see [6, 92, 59].

The strategy to study Kähler-Einstein metrics on a smooth Fano manifold with cone singularities along a smooth divisor, witch is proportional to the anti-canonical divisor, is "interpolation-degeneration". By "interpolation" we show the angles in $(0, 2\pi]$ that admit a conical Kähler-Einstein metric form an interval, and by "degeneration" we figure out the boundary of the interval [60].

5.4 KE metric on Fano manifold

The following remarkable result was proved by Chen, Donaldson and Sun [29, 30, 31].

Theorem 5.3 (Chen-Donaldson-Sun)**.** *A Fano manifold X is K-stable if and only if it admit a Kähler-Einstein metric.*

The strategy to prove the existence of KE on Fano manifold follows that suggested in [43]. We fix some $\lambda > 0$ such that the linear system $|\lambda K_X|$ contains a smooth divisor D. We consider the Kähler-Einstein metrics on X with a cone singularity of cone angle β along D, where $\beta \in (0, 1]$ is a variable parameter. Of course when $\beta = 1$ these are just smooth Kähler-Einstein metrics. Such metrics with cone singularities were discussed in general terms some years ago by Tian [97]. More recently, following [43], a detailed theory has been developed, both on the differential geometric side [11, 43, 44, 85] and the algebraic-geometric side [60, 72, 88].

The fundamental point is that the modified Futaki invariant $Fut_\beta(\mathcal{X})$ is linear in β, so if we know that $Fut_\beta(\mathcal{X}) = 0$ for some $\beta \leqslant 1$ and we know that $Fut_{\beta'}(\mathcal{X}) \geqslant 0$ for some $\beta' < \beta$ then we can deduce that $Fut(\mathcal{X}) \leqslant 0$ and so X is not K-stable. The existence of a Kähler-Einstein metric with small cone angle is well-understood. So the problem becomes to show that if we have an increasing sequence $\beta_i \to \beta_\infty$, with $\beta_\infty \leqslant 1$ such that there are Kähler-Einstein metrics ω_i with these angles then either there is a Kähler-Einstein metric with cone angle β_∞ or there is a test configuration X with $Fut_{\beta_\infty}(\mathcal{X})$. Same remark as above about the small complication when X has automorphisms. This strategy can be regarded as a variant of the standard "continuity method", in which one perturbs the Kähler-Einstein equation by using a positive (1,1) form.

5.5 Stable pair

Let G be one of the classical subgroups of $GL(N+1, \mathbb{C})$, (V, ρ) be a finite dimensional complex rational representation of G. Let H denote any maximal algebraic torus of G. By Peter-Weyl theorem and Fourier analysis on Abelian Lie group, algebraic homorphism $\lambda : H \to \mathbb{C}^*$ form a lattice in $\mathbb{R}^N$, we denote the lattice and its dual by $M_{\mathbb{Z}}, N_{\mathbb{Z}}$. Using standard representation theory, we can decompose V into the weight spaces

$$V = \bigoplus_{\lambda \in supp(V)} V_\lambda. \tag{5.3}$$

For $v \in V \backslash \{0\}$, we define the weight polytope of v is the compact convex integral polytope $\mathcal{N}(v)$ given by the convex hull of the lattice points $\{\lambda \in supp(v)\}$. The reader who is not familiar with representation of Lie group can take [12] as a reference.

Definition 5.1. *Let V and W be finite dimensional complex rational representations of G. Let $v \in V \backslash \{0\}$ and $w \in W \backslash \{0\}$. The pair (v, w) is K-semistable if and only if for all H, $\mathcal{N}(v) \subset \mathcal{N}(w)$. Pair (v, w) is K-stable if and only if for all H, there exist $m_0 \in \mathbb{N}$ such that $(v^{m-1} \otimes u^{q(V)}, w^m)$ is K-semistable for all $m \geqslant m_0$, where u is any H-generic vector and $q(V)$ is degree in the standard representation of G.*

Remark 5.1. *The reader may easily verify that Hilbert-Mumford stability is a special case of K-stability. In particular, it provides many examples of K-semistable pairs.*

A nontrivial special case of K-stability arises in connection with complex projective varieties. In order to proceed, let us first recall the Hilbert-Mumford stability theory. The core of this theory consists of associating a vector bundle E over a subvariety $X \to \mathbb{P}^N$, a projective geometric gadget that encodes the object up to projective equivalence. More precisely, one associates these data with an orbit $G \cdot v$ of some nonzero vector v in a finite dimensional complex rational G module E. For example, one associates the Gieseker point to $E \to X$, and one associates either the Hilbert point or the Chow form to a subvariety $X \to \mathbb{P}^N$. Similarly, in order to apply K-stability to a smooth projective variety $X \to \mathbb{P}^N$ we must associate our embedded variety X to a pair $v(X) \in V \backslash \{0\}$, $w(X) \in W \backslash \{0\}$, where V and W are finite dimensional rational G-representations. The notation is intended to suggest that X is encoded by the pair (v, w). As the reader shall see, each vector is projectively natural.

Let $X^n \to \mathbb{P}^N$ be an irreducible, linearly normal subvariety of degree d. The Cayley-Chow form of X, denoted by R_X, is the defining polynomial (unique up to scaling) of the divisor

$$\left\{ L \in G\left(N-n, \mathbb{C}^{N+1}\right) \,\middle|\, L \cap X \neq \varnothing \right\} = \left\{ L \,\middle|\, R_X(L) = 0 \right\}. \tag{5.4}$$

When the dual variety $X^\vee$ is indeed a hypersurface, we have the defining polynomial Δ_X, unique modulo scaling. Just as in the case of classical resultants and discriminants of polynomials in one variable, we may view the general X-discriminant and Cayley-Chow form as homogeneous polynomials on spaces of matrices: $\Delta_X \in \mathbb{C}\left[M_{1\times(N+1)}\right]$, $R_X \in \mathbb{C}\left[M_{(n+1)\times(N+1)}\right]$.

Definition 5.2. *We call that X is K-semistable (Definition 5.1) if and only if the pair $\left(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)}\right)$ is K-semistable for the action of G; X is K-stable if and only if the pair $\left(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)}\right)$ is K-stable for the action of G.*

Paul proved that the Mabuchi energy of $\left(X, \omega_{FS}|_X\right)$ restricted to the Bergman metrics is completely determined by the X-hyperdiscriminant of format $(n-1)$ and the Chow form of X. As a corollary, it is shown that the Mabuchi energy is bounded from below for all degenerations in G if and only if the hyperdiscriminant polytope dominates the Chow polytope for all maximal algebraic tori H of G.

Theorem 5.4. *Let $X^n \to \mathbb{P}^N$ be a smooth, linearly normal, complex algebraic variety of degree $d \geqslant 2$. Then there are norms such that the Mabuchi energy restricted to the Bergman metrics is given as follows*

$$\nu_\omega(\varphi_\sigma) = \deg(R_X) \log \frac{\|\sigma \cdot \Delta_{X\times\mathbb{P}^{n-1}}\|^2}{\|\Delta_{X\times\mathbb{P}^{n-1}}\|^2} - \deg(\Delta_{X\times\mathbb{P}^{n-1}}) \log \frac{\|\sigma \cdot R_X\|^2}{\|R_X\|^2}. \tag{5.5}$$

The norms which appear on the right-hand side are conformally equivalent to the standard norms on the spaces of polynomials respectively. These norms were first constructed in [98]. With this theorem in mind, we reduce the problem of bounding the Mabuchi energy from below to analyzing the simultaneous G orbit of the resultant and hyperdiscriminant polynomials inside certain irreducible G modules $S_{\lambda.}(\mathbb{C}^{N+1})$ and $S_{\mu.}(\mathbb{C}^{N+1})$ respectively. We are now prepared to state the fundamental corollary, which first completely provides the algebraic characterization of the existence of the lower bound for the Mabuchi energy on the space of Bergman metrics.

Corollary 5.1. *The Mabuchi energy is bounded below if and only if*

$$\overline{G\cdot[(R^{\deg(\Delta)},\Delta^{\deg(R)})]}\bigcap\overline{G\cdot[(R^{\deg(\Delta)},0)]}=\varnothing. \tag{5.6}$$

Remark 5.2. *It follows from this corollary that the asymptotic expansion of the Mabuchi energy along any algebraic one parameter subgroup of H (a maximal algebraic torus of G) is completely determined by the Chow polytope $\mathcal{N}(R_X)$ and the hyperdiscriminant polytope $\mathcal{N}(\Delta_{X\times\mathbb{P}^{n-1}})$. As $t\to 0$, we have*

$$\mu(\lambda(t))=F_P(\lambda)\log|t|^2+O(1), \tag{5.7}$$

where

$$F_P(\lambda):=\deg(R_X)\min_{x\in\mathcal{N}(\Delta_{X\times\mathbb{P}^{n-1}})}l_\lambda(x)-\deg(\Delta_{X\times\mathbb{P}^{n-1}})\min_{x\in\mathcal{N}(R_X)}l_\lambda(x). \tag{5.8}$$

This gives a complete description of the behavior of the Mabuchi energy along all degenerations, that $\mu(\lambda(t))$ has a logarithmic singularity as $t\to 0$ and the coefficient of blow up is an integer.

5.6 Analytic stability of cscK

With the existence problem of Kähler-Einstein metric settled eventually, it is time to discuss how to attack Calabi's original problem in full generality. The cscK metric equation can be rewritten as a pair of coupled equations

$$\begin{aligned}\log\det(g_{i\bar{j}}+\varphi_{i\bar{j}})&=F+\log\det(g_{i\bar{j}}),\\ \Delta_\varphi F&=-\underline{R}+tr_\varphi Ric_g.\end{aligned} \tag{5.9}$$

To attack the existence problem of cscK metrics, we have to study a 4th order PDE as above. Chen [24] proposed a "new" continuity path in a given Kähler class to solve the cscK metric problem. Module out the profound difficulty in analysis, this idea shed light on the existence problem from direct PDE approach.

For any positive, closed (1,1)-form χ, we define a continuous path $t\in[0,1]$ as

$$t\left(R_{\varphi_t}-\frac{[c_1(M)][\omega_0]^{[n-1]}}{[\omega_0]^{[n]}}\right)=(1-t)\left(tr_{\varphi_t}\chi-\frac{[\chi][\omega_0]^{[n-1]}}{[\omega_0]^{[n]}}\right). \tag{5.10}$$

A Kähler metric is called twisted cscK metric if its scalar curvature satisfies the equation above. We call it twisted extremal Kahler metric if the left hand side of the Equation gives rise to a holomorphic vector field.

When $t=1$, this reduces to the equation for cscK metrics. Let I denote the set of time parameter $t \in [0,1]$ such that the equation can be solved at time t. As usual, our goal is to first prove that I is not empty which usually means finding a starting point where we can solve this equation. Then, we prove I is open which is crucial for this program to be viable. The hard part is of course to prove I is closed which involves a difficult a priori estimate.

The conspicuous and memorable feature of CDS's proof is the heavy use of Cheeger-Colding theory on manifold with Ricci curvature bounded from below. The a priori bound on Ricci curvature for KE metrics made such an application of Cheeger-Colding theory seamlessly smooth and effective. However, if we want to attack this general conjecture, there will be a dauntingly high wall to climb since there is no a priori bound on Ricci curvature. Therefore, the entire Cheeger-Colding theory needs to be re-developed if it is at all feasible. On the other hand, there is a second, less visible but perhaps even more significant feature of CDS's proof: The whole proof is designed for constant scalar curvature Kähler metrics, and the use of algebraic criteria and Cheeger-Colding theory is to conclude that the a C^0 bound holds for Kähler potential so that we apply the apriori estimates for complex KE metrics developed by Calabi, Yau and others. Indeed, this is exactly how Chen and Cheng made use of Cheeger Colding theory and stability condition in CDS's proof to nail down a C^0 estimate on potential. Unfortunately, such an estimate is missing in this generality for a 4th order fully nonlinear equation. Indeed, as noted by other famous authors in the subject as well, the difficulty permeates the cscK theory are two folds: one cannot use maximal principle from PDE point of view and one can not have much control of metric from the bound of the scalar curvature. Recently, Chen and Cheng have an estimate about cscK [25], which makes Yau-Tian-Donaldson conjecture probably resolvable.

Theorem 5.5. *If* (M, ω_φ) *is a cscK metric, then all higher derivatives of the Kähler potential function* φ *can be estimated in terms of upper bounded of* $\int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\omega_\varphi^n$.

As a consequence, it shows that all higher derivatives of φ can be estimated in terms of $\|\varphi\|_0$. This estimate gives an equivalence between cscK metric and analytic stability [26, 27].

Theorem 5.6 (Chen-Cheng)**.** *The polarized Kähler manifold* $(M, [\omega_0])$ *is geodesic stable if and only if the Kähler class* $[\omega_0]$ *admits a cscK metric.*

5.7 Completion of the proof of Yau-Tian-Donaldson conjecture.

Now we are in a position to give a complete proof of YTD conjecture. Given a homogeneous degree d polynomial P on $\mathbb{C}^{N+1}$, we identify it with a section of canonical bundle $O(d)$ over $\mathbb{CP}^N$. For any $p \in [0, \infty]$ we define the L^p norm by using the Fubini-Study metric. When $p = 0$, it is defined by

$$\log \|P\|_0 := \int_{\mathbb{CP}^N} \log |P|_{h_{FS}^d} \omega_{FS}^N, \tag{5.11}$$

where $\log\|P\|_0$ is called the logarithmic Mahler measure of P. By a well-known proporsition [10, 46, 58], we have

$$-\frac{d}{2}\left(\sum_{j=1}^{N}\frac{1}{j}\right) + \log\|P\|_p \leqslant \log\|P\|_0 \leqslant \log\|P\|_p. \tag{5.12}$$

We assume that X is smooth and linearly normal, Choose any L^p normalized R, Δ, then the L^p distance between the points $\left(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)}\right)$ and $\left(R_X^{\deg(\Delta_X)}, 0\right)$ is defined by

$$\log\tan\operatorname{dist}_p(\sigma) := \log\|\sigma\cdot\Delta_X^{\deg(R_X)}\|_p - \log\|\sigma\cdot R_X^{\deg(\Delta_X)}\|_p. \tag{5.13}$$

The L^p distance between the orbit closures is defined to be

$$\log\tan\operatorname{dist}_p\left(\overline{\mathcal{O}_{R\Delta}}, \overline{\mathcal{O}_R}\right) := \inf_{\sigma\in G}\log\tan\operatorname{dist}_p(\sigma). \tag{5.14}$$

The point is that all of the L^p distances measure the same thing: any one of them detects the semistability of X. What is extraordinary is that the infimum of the Mabuchi energy restricted to the Bergman metrics at level k is exactly the distance between the orbit closures in the L_0 distance.

Definition 5.3. *A polarized manifold (X, L) is asymptotically semistable if and only if there is a uniform constant $C = C(h) \geqslant 0$ such that*

$$\operatorname{dist}_0\left(\overline{\mathcal{O}_{R\Delta}}, \overline{\mathcal{O}_R}\right) \gtrsim e^{-Cd^2} \tag{5.15}$$

for all sufficiently large L^k-embeddings of degree $d = k^n$. A polarized manifold (X, L) is asymptotically stable if and only if there are uniform constant $m \in \mathbb{Z}^+$ and $C = C(h, m) \geqslant 0$ such that

$$\operatorname{dist}_0\left(\overline{\mathcal{O}_{(v,w)}}, \overline{\mathcal{O}_v}\right) \gtrsim e^{-Ck^{2n+1}} \tag{5.16}$$

for all sufficiently large k (the power of the embedding). Where

$$(v, w) := \left(I^q \otimes R_X^{(km-1)\deg(\Delta_X)}, \Delta_X^{km\deg(R_X)}\right). \tag{5.17}$$

As in the definition of the asymptotic semistability, both R_X and Δ_X have been scaled to have length one in the norm $\|\cdot\|_0$. One should observe that the rates of convergence to the orbit closures in the definitions, the asymptotic stability, and the semistability differ by a single factor of k.

The norm appearing in Theorem 5.4 was first considered by Tian in his early works on CM stability [98, 99, 100, 101, 102]. This norm is conformally equivalent to the L^2 norm with a continuous potential $\|\cdot\| := e^{\theta}\|\cdot\|_{L^2}$. It seems there is very little that one could say about θ beyond its Hölder continuity. However, for families of divisors, the situation considered here, θ can be described explicitly, which allows us to significantly improve Theorem 5.4.

$$\nu_\omega(\varphi_\sigma) = \deg(R_X)\log\frac{\|\sigma\cdot\Delta_{X\times\mathbb{P}^{n-1}}\|_0^2}{\|\Delta_{X\times\mathbb{P}^{n-1}}\|_0^2} - \deg(\Delta_{X\times\mathbb{P}^{n-1}})\log\frac{\|\sigma\cdot R_X\|_0^2}{\|R_X\|_0^2}. \tag{5.18}$$

We identify the space of homogeneous polynomials of degree d with a section of canonical bundle $O(d)$ over $\mathbb{CP}^N$. Then we let B denote the corresponding complete linear system and $\mathscr{X}_d$ the universal family of hypersurfaces over B. An explicit description of θ is obtained by noting that $\mathscr{X}_d$ is a divisor in $B \times \mathbb{P}^{n+1}$ cut out by a section Ψ of $p_1^*\mathcal{O}_B(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^n}(d)$. The following crucial observation was shown by Tian.

Lemma 5.1. *There is a uniform constant C such that for all sufficiently large $k \in \mathbb{N}$ we* *have*

$$C + \frac{1}{k}\log\left(\frac{\|\sigma\|^2}{N_k+1}\right) \leqslant \int_X \frac{\Psi_\sigma}{k}\frac{\omega^n}{V_0}. \tag{5.19}$$

We compare the Mabuchi and Aubin energies of the reference metric with the restrictions of the Fubini-Study metrics coming from the large projective embeddings, then we have

$$J_\omega\left(\frac{\Psi_\sigma}{k}\right) = \frac{1}{k}J_{\omega_{FS}|_{l_k(X)}}(\varphi_\sigma) + o(1). \tag{5.20}$$

The comparison formulas (5.20) and the lemma 5.1 imply that

$$J_\omega\left(\frac{\Psi_\sigma}{k}\right) = \frac{1}{k}F^0_{\omega_{FS}|_{l_k(X)}}(\varphi_\sigma) + \frac{1}{k}\log\|\sigma\|^2 + O(1). \tag{5.21}$$

Recall a well known proposition [78, 86]

$$-\deg(R_X)F^0_{\omega_{FS}|_{l_k(X)}}(\varphi_\sigma) = \log\|\sigma\cdot R_X\|_0. \tag{5.22}$$

We have chosen R_X to have length one in the Mahler norm. Inserting (5.21) into (5.22) allows us to express J_{ω_h} restricted to Bergman kernel as a distance function

$$\frac{\deg(\Delta_X)}{d}J_{\omega_h}\left(\frac{\Psi_\sigma}{k}\right) = \frac{1}{k^{2n+1}(n+1)}\left(-\deg(\Delta_X)\log\|\sigma\cdot R_X\|_0^2 + q\log\|\sigma\|^2\right) + O(1). \tag{5.23}$$

The equation (5.18) and comparison formulas (5.20) give

$$m\nu_{\omega_h}\left(\frac{\Psi_\sigma}{k}\right) = \frac{-m\deg(\Delta_X)\log\|\sigma\cdot R_X\|_0^2 + m\deg(R_X)\log\|\sigma\cdot\Delta_X\|_0^2}{k^{2n}(n+1)} + o(1). \tag{5.24}$$

As usual, we have chosen representatives satisfying $\|R_X\|_0 = \|\Delta_X\|_0 = 1$. Subtract (5.23) from (5.24) and use the definition of the L^0 distance to get

$$m\nu_{\omega_h}\left(\frac{\Psi_\sigma}{k}\right) - \frac{\deg(\Delta_X)}{d}J_{\omega_h}\left(\frac{\Psi_\sigma}{k}\right) = \frac{\log\tan\mathrm{dist}_0(\sigma\cdot[(v,w)])}{k^{2n+1}(n+1)} + O(1). \tag{5.25}$$

Recall that the pair (v, w) is given by (5.17), taking the inf over G on both sides of (5.25), we get

$$\inf_{\frac{\Psi_\sigma}{k}\in\mathscr{B}_{N_k}}\left(m\nu_{\omega_h}\left(\frac{\Psi_\sigma}{k}\right) - \frac{\deg(\Delta_X)}{d}J_{\omega_h}\left(\frac{\Psi_\sigma}{k}\right)\right) = \frac{\log\tan\mathrm{dist}_0\left(\overline{O_{(v,w)}}, \overline{O_v}\right)}{k^{2n+1}(n+1)} + O(1). \tag{5.26}$$

Now we complete the proof of Yau-Tian-Donaldson conjecture 1.1.

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作时间为信笺

行走大地成笔墨

刻下生命的诗篇

笔墨诗篇

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——《西江月 · 夜行黄沙道中》

诗词四首

吴叩天

七律·嫦五升空时评

闻嫦五成功升空有感，仿李鸿章《入都》而记之。

昔闻太白道玉钩①，今上蟾宫九重楼。
五千年来窥宝鉴②，百万里外③望神州。
玉兔当先④稳步落⑤，嫦娥殿后大功收⑥。
撷得月壤⑦凯旋日，笑看深蓝小寰球⑧。

行香子·十八自寿

时九月十一夜，庐州秋风秋雨，与友欢庆成年。作此篇，不谏已往，以追来者。

衰草蛩鸣，清露霖铃。
楚天暝、秋水盈盈。
南柯梦后，且忘芳卿。
作可心诗，舒心画，会心声。

忽闻雨霁，夜已三更。
象牙塔、浩荡今生。
书生意气，赤子豪情。
但创新篇，开新路，赴新程。

天净沙·五一

2021 年 5 月 1 日

昂首烈日晴空，
迎面桐絮东风，
转头落花春梦。
舒卷执笔，
暂避心锁樊笼。

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稿件信息: 3 月 3 日来稿，3 月 20 日第一次修改，3 月 21 日接受。稿件编辑: 黄欣祺。

① 玉钩：李白《挂席江上待月有怀》：“倏忽城西郭，青天悬玉钩。”
② 宝鉴：欧阳修《渔家傲·其八》：“池上月华开宝鉴。”
③ 百万里外：月球与地球远地点的距离是 40.6 万千米，约为一百万里。
④ 玉兔当先：玉兔号是中国首辆月球车，和着陆器共同组成嫦娥三号探测器。玉兔号也是中国在月球上留下的第一个足迹，意义深远。
⑤ 稳步落：中国探月工程采用“绕、落、回”三步走发展战略。我国探月工程的第二步是“落”月探测，其中嫦娥三号和嫦娥四号实现“落”月。
⑥ 嫦娥殿后大功收：“嫦娥五号”将执行我国探月三期任务“采样返回”。探月三期的目标是实现我国首次月面自动采样返回，对返回样品进行系统分析与研究，深化对月球和地月系统的起源与演化的认识。这也将为载人登月和深空探测奠定基础。嫦娥 5 号成功发射奔月，是“绕、落、回”战略的“收官之作”。
⑦ 撷得月壤：“嫦娥五号”将执行月壤采样任务。
⑧ 深蓝小寰球：指从月球上看，地球就像一个蓝色小球一样。毛泽东《满江红·和郭沫若同志》：“小小寰球，有几个苍蝇碰壁。”

行香子·马拉松

2021 年 11 月 14 日，科大英才班秋季马拉松。走遍全程，神清气爽。作此篇，兼赠刘君。

赤日林梢。山径岧峣①。
马拉松道共华韶。
熙熙黄发，攘攘垂髫②。
看前者奔，后者走，众逍遥。
既观胜境，又品佳肴。
难平意恨竟全消。
人生海海，世事潮潮。
望云轻轻，天阔阔，路迢迢。

① 岧峣：亦作“迢峣”。山高峻貌。
② 黄发垂髫：不少老老师，还有不少老师带小孩。

浣溪沙九首

余淼

浣溪沙·茗菊

飒飒寒珠跳玉阶，东篱香浅碧橙斜，采撷浸水两三些。
浮蕊飞烟留质态，近窗招取冷花蝶，一酌入梦早寒歇。

浣溪沙·新凉

圆露流烟草色寒，朝来微醒漫寥天。着衣掠洗试新餐。
大抵衷情如柳叶，北风零乱悴①华颜。厌花畏酒只春眠。

浣溪沙·转凉

清晓揭窗陡觉寒，薄衾凉簟奈衣单，心惊秋气落人间。
碧玉渐将成琥珀，啼红染雨信风眠。斯时方感物华妍。

浣溪沙·无题

镜里凉秋似那年，东城闲步晚来寒，黄花风下正流连。
双月慵慵无聊赖，轻霜偏把心事牵，燕台老句②系钗钿③。

浣溪沙·思锦官

岂料庐阳夏日忙，炎天錾④碧树花黄，闻说锦官正秋凉。
几日已然思旧景，爽风杏叶入尘窗。芙蓉香里梦魂长。

浣溪沙·闲作

洗沐衣轻染袂香，闲敲棋子放勤忙，酸橙新⑤破指间凉。
算是明朝应早起，旋思读写已结央。稍稍可以晚赖床。

浣溪沙·自锦官归庐阳

天角遮乌复冷清，惺忪梦里赴航程，转头一月黯然惊。
仆仆风尘三万里，离乡总怯忘乡音，待春归后把春寻。

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稿件信息: 3 月 1 日来稿，3 月 23 日第一次修改，3 月 24 日接受。稿件编辑: 黄欣祺。

① 悴：使憔悴。

② 燕台老句：“燕台句”即工于言情的诗句，所谓“句老”是一来说这些句子写来有很久了，二是写不出什么新意。

③ 系钗钿：“钗钿”借女子的首饰指代所忆的佳人。前句所牵的心事亦在于此。

④ 錾：在金石上雕刻，“錾碧”是为碧叶叶色鲜艳明丽，仰望而去，如同刻在天幕之中。

⑤ 新：即“才，刚刚”之意。

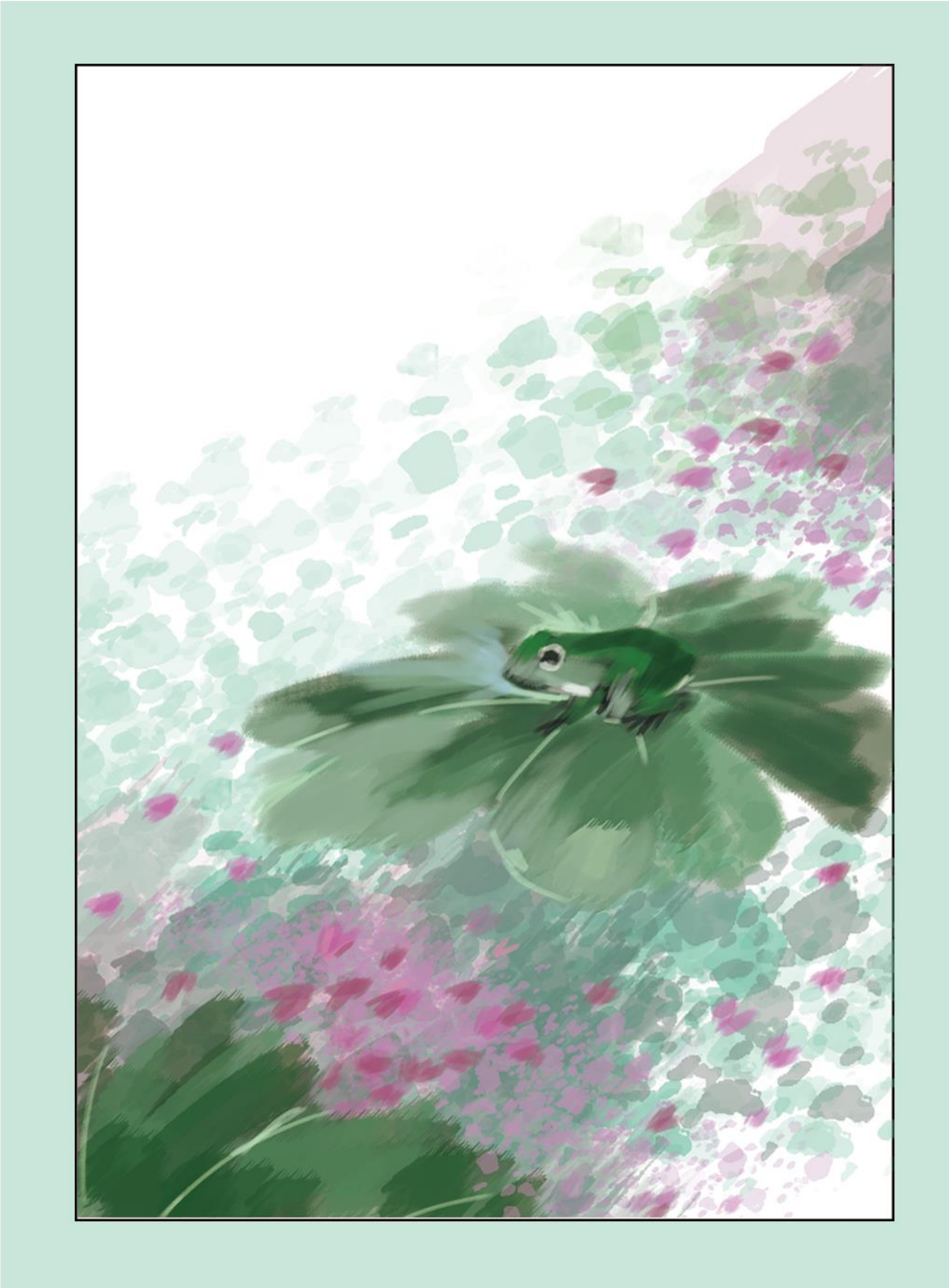
浣溪沙·初暖

初暖重游往日池，小桥薄雪嫩桃枝，水边早梅怪春迟。
软水新消犹沁冷，蜂蝶莺燕几未齐，何妨细履步芳时？

浣溪沙·春早

春早寒清暖照轻，光枝遒劲有龙吟，巧听几点鹧鸪声。
闻道西南春更早，东南海燕寄梅音，总对荷藕话春晴。

皇榜



稻花香里说丰年，听取蛙声一片。

——《西江月 · 夜行黄沙道中》

皇榜第二期题目解答

第 1 题

计算

$$\int_0^{\frac{\pi}{2}} \frac{\{\cot(x)\}}{\cot(x)} dx$$

其中 $\{x\}$ 表示 x 的小数部分.

解答:

注意到 $\{\cot x\}$ 有界, $\cot x$ 在 $(0,\pi)$ 上单调, 且当 $x \to 0$ 时 $\cot x \to 0$, 由 Dirichlet 判别法知原积分收敛. 下面来计算它的积分值, 为此, 首先给出一个大家在复分析中熟知的引理: 对任何 $z \in \mathbb{C}$, 有

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

回到原题. 令 $[x]$ 表示 x 的整数部分, 则

$$\int_0^{\frac{\pi}{2}} \frac{\{\cot x\}}{\cot x} = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{[\cot x]}{\cot x} = \frac{\pi}{2} - \sum_{n=0}^{\infty} \int_{\operatorname{arccot}(n+1)}^{\operatorname{arccot} n} \frac{[\cot x]}{\cot x} \mathrm{d}x.$$

而

$$\int_{\operatorname{arccot}(n+1)}^{\operatorname{arccot} n} \frac{[\cot x]}{\cot x} \mathrm{d}x = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{nt}{1+t^2} \mathrm{d}t = n(a_n - a_{n+1}),$$

其中 $a_n := \frac{1}{2} \ln\left(1 + \frac{1}{n^2}\right)$. 注意到 $na_n \to 0$ 以及 a_n 构成的级数收敛, 我们有

$$\int_0^{\frac{\pi}{2}} \frac{\{\cot x\}}{\cot x} = \frac{\pi}{2} - \sum_{n=1}^{\infty} a_n.$$

应用引理我们就得到了

$$\int_0^{\frac{\pi}{2}} \frac{\{\cot x\}}{\cot x} = \frac{\pi}{2} - \frac{1}{2} \ln\left[\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)\right] = \frac{\pi}{2} - \frac{1}{2} \ln \frac{e^{\pi} - e^{-\pi}}{2\pi}.$$

□

编者按: 此解答来自董凯同学, 另外吴天, 王乐达, 罗云杰也给出了类似的解答, 在此不赘. 值得一提的是, 王乐达同学在计算级数 $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ 时采取了另一公式 $|\Gamma(a+bi)|^2 = |\Gamma(a)|^2/\prod_{k=0}^{\infty}(1+\frac{b^2}{(a+k)^2})$, 他称之为"Legendre duplication formula 的一种推广形式".

第 2 题

证明: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n-1)16^n} \binom{2n}{n}^2 \sum_{k=n}^{\infty} \frac{2^k}{k\binom{2k}{k}} = 1 - \sqrt{2} + \log(1+\sqrt{2}).$

解答:

记所求积分为 I, 由 $2t(1-t) \leqslant \frac{1}{2}$, $\forall t \in [0,1]$, 结合 Weierstrass 判别法, 知:

$$\sum_{k=n}^{\infty} \frac{2^k}{k\binom{2k}{k}} = \sum_{k=n}^{\infty} 2^k \mathrm{B}(k+1,k) = \int_0^1 \sum_{k=n}^{\infty} (2t)^k (1-t)^{k-1} \mathrm{d}t = \int_0^1 \frac{(2t-2t^2)^n}{(2t^2-2t+1)(1-t)} \mathrm{d}t.$$

注意到 $\dfrac{n}{(2n-1)16^n}\dbinom{2n}{n}=\dfrac{1}{2}\dfrac{(2n-1)!!(2n-3)!!}{(2n)!!(2n-2)!!}$, 及 $\sum\limits_{n=0}^{\infty}\dfrac{(2n-1)!!}{(2n)!!}x^{2n}$ 收敛, $\forall|x|<1$, 有:

$$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{1-x^2\cos^2\theta}}=\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x^2\cos^2\theta)^n\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}\cos^{2n}\theta\mathrm{d}\theta$$

$$=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}\cdot\frac{1}{2}\mathrm{B}(n+\frac{1}{2},\frac{1}{2})=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(2n-1)!!^2}{(2n)!!^2}x^{2n},\quad\forall x\in(-1,1).$$

进而对 $\forall|x|<1$,

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!(2n-3)!!}{(2n)!!(2n-2)!!}x^{2n}=x\int_0^x\frac{1}{y}\left(\sum_{n=0}^{\infty}\frac{(2n-1)!!^2}{(2n)!!^2}y^{2n}\right)'\mathrm{d}y=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{x^2\cos^2\theta}{\sqrt{1-x^2\cos^2\theta}}\mathrm{d}\theta.$$

因此 $I=\dfrac{2}{\pi}\displaystyle\int_0^1\frac{t\mathrm{d}t}{2t^2-2t+1}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\mathrm{d}\theta}{\sqrt{1+(2t^2-2t)\cos^2\theta}}$.

分 $[0,\frac{1}{2}]$ 与 $[\frac{1}{2},1]$ 两段, 考虑 t 换成 $1-t$ 换元, 得:

$$\begin{aligned}I&=\frac{2}{\pi}\int_0^{\frac{1}{2}}\frac{\mathrm{d}t}{2t^2-2t+1}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\mathrm{d}\theta}{\sqrt{1+(2t^2-2t)\cos^2\theta}}\\&\xlongequal{t\to\frac{1}{2}-t}\frac{2}{\pi}\int_0^{\frac{1}{2}}\frac{\mathrm{d}t}{2t^2+\frac{1}{2}}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\mathrm{d}\theta}{\sqrt{(2t^2+\frac{1}{2})\cos^2\theta+\sin^2\theta}}.\end{aligned}$$

当 $a,b>0$ 时, 考虑 $u=\dfrac{t}{\sqrt{(2t^2+\frac{1}{2})a^2+b^2}}$, $\mathrm{d}u=\dfrac{(b^2+\frac{a^2}{2})\mathrm{d}t}{\left((2t^2+\frac{1}{2})a^2+b^2\right)^{\frac{3}{2}}}$, 研究不定积分:

$$\int\frac{\mathrm{d}t}{(2t^2+\frac{1}{2})\sqrt{(2t^2+\frac{1}{2})a^2+b^2}}=\int\frac{\mathrm{d}u}{2b^2u+\frac{1}{2}}=\frac{1}{b}\arctan\frac{2bt}{\sqrt{(2t^2+\frac{1}{2})a^2+b^2}}+C,$$

令 $a=\cos\theta$, $b=\sin\theta$, 再次结合一致收敛和 Beta 函数的性质, 有:

$$I=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\arctan\sin\theta}{\sin\theta}\mathrm{d}\theta=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\cos^2\theta\mathrm{d}\theta=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\mathrm{B}(n+\frac{1}{2},\frac{3}{2})$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n+1)(2n+2)!!}=\int_0^1\left(\int_0^x\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n)!!}y^{2n}\mathrm{d}y\right)\mathrm{d}x=\int_0^1\left(\int_0^x\frac{1}{\sqrt{1+y^2}}\mathrm{d}y\right)\mathrm{d}x$$

$$=\int_0^1\log(x+\sqrt{1+x^2})\mathrm{d}x=\log(1+\sqrt{2})-\int_0^1\frac{x}{\sqrt{1+x^2}}\mathrm{d}x=1-\sqrt{2}+\log(1+\sqrt{2}).\qquad\square$$

编者按: 此解答来自吴天同学, 他熟练地运用了特殊函数积分与级数之间的关系, 透露出了较强的分析功夫.

第 3 题

设 $\Omega\subset\mathbb{R}^2$ 为一个区域且 $u(x,t)\in L^1_{loc}(\Omega\times(0,1)\setminus\Gamma)$ 为以下方程的一个非负解

$$\frac{\partial u}{\partial t}=\Delta u,\quad\forall(x,t)\in\big(\Omega\times(0,1)\big)\setminus\Gamma,$$

其中 $\Gamma=\{(\xi(t),t)|t\in(0,1)\}$. 且 $\xi:[0,1]\to\Omega$ 是一条 α-Holder 连续曲线.

(1) 若 $\alpha \in (\frac{1}{2}, 1)$. 证明: $u \in L^1_{loc}\big(\Omega \times (0,1)\big)$.

(2) 若 $\alpha \in (0, \frac{1}{2}]$, (1) 的结论还成立吗?

(3) 设 $\Gamma = \{(\xi_1(t), t) | t \in (0,1)\} \cup \{(\xi_2(t), t) | t \in (0,1)\}$, 其中 $\xi_i : [0,1] \to \Omega (i = 1, 2)$ 为 α-Holder 连续曲线且 $\alpha \in (\frac{1}{2}, 1)$, (1) 的结论还成立吗?

编者按: 此题目前无人给出解答, 欢迎各位继续投稿.

第 4 题

设 n 为给定的正整数, $P(x)(x \in \mathbb{R})$ 为实系数多项式

$$P(x) = x^m + 2x^{m-1} + \cdots + mx + (m+1),$$

求尽可能小的实数 x_0 的值, 使得 $P(x)$ 的任意阶导数在 x_0 处均非负.

解答: 注意到 $P^{(m-1)}(x) = m!x + 2(m-1)!$, 故 $x_0 \geq -\frac{2}{m}$.

下证 $x_0 = -\frac{2}{m}$ 时, $P^{(i)}(x_0) \geq 0 (\forall 0 \leq i \leq m)$. 从而 $x_0 = -\frac{2}{m}$ 即所求值.

事实上,

$$P^{(i)}(x) = \sum_{p=0}^{m-i} (m-i+1-p)(i+p)(i+p-1)\cdots(p+1)x^p$$

为方便书写, 记 $a(p,i) = (i+p)(i+p-1)\cdots(p+1)$.

若 $m-i$ 为偶数, 则

$$P^{(i)}(x) = a(m-i,i)x^{m-i} + \sum_{p=0}^{\frac{m-i}{2}-1} \big((m-i+1-2p)a(2p,i)x^{2p} + (m-i-2p)a(2p+1,i)x^{2p+1}\big). \tag{1}$$

若 $m-i$ 为奇数, 则

$$P^{(i)}(x) = \sum_{p=0}^{\frac{m-i-1}{2}} \big((m-i+1-2p)a(2p,i)x^{2p} + (m-i-2p)a(2p+1,i)x^{2p+1}\big) \tag{2}$$

注意 $P^{(m-1)}\left(-\frac{2}{m}\right) = 0$, 故可设 $i \leq m-2$. 由于待证命题在 $1 \leq m \leq 10$ 时是平凡的, 故设 $m > 10$. 为方便书写, 记

$$A_p(x) = (m-i+1-2p)a(2p,i)x^{2p} + (m-i-2p)a(2p+1,i)x^{2p+1}$$

断言 1: 若 $1 \leq p \leq \frac{m-i-1}{2}$, 则 $A_p\left(-\frac{2}{m}\right) > 0$.

证明: 只需 $\frac{(m-i+1-2p)}{m-i-2p}\frac{m(2p+1)}{2(i+2p+1)} > 1$ 由于 $1 \leq p \leq \frac{m-i-1}{2}$, 并且 $\frac{m(2p+1)}{2(i+2p+1)} \geq \frac{3m}{2m} > 1$ 故断言 1 得证.

由断言 1, (1)(2) 的右边中除了 $A_0\left(-\frac{2}{m}\right)$ 均是正的. 注意到, 当 $0 \leq i \leq \frac{m}{2}$ 时,

$$\begin{aligned} A_0\left(-\frac{2}{m}\right) &= (m-i)(i+1)\cdots 2\left(-\frac{2}{m}\right) + (m-i+1)i(i-1)\cdots 1 \\ &= \frac{i!}{m}(m-2i)(m-i-1) \\ &\geq 0. \end{aligned}$$

下面只需处理 $i > \frac{m}{2}$ 的情形.

断言 2: 当 $\frac{1}{2}m < i \leq m-5$ 时, $(A_0 + A_1 + A_2)\left(-\frac{2}{m}\right) > 0$.

证明: 事实上,

$$
\begin{aligned}
&\frac{120}{i!}\left(A_0+A_1+A_2\right)\left(-\frac{2}{m}\right)\\
=&-\frac{32}{m^5}(m-i-4)(i+5)(i+4)(i+3)(i+2)(i+1)\\
&+\frac{80}{m^4}(m-i-3)(i+4)(i+3)(i+2)(i+1)\\
&-\frac{160}{m^3}(m-i-2)(i+3)(i+2)(i+1)+\frac{240}{m^2}(m-i-1)(i+2)(i+1)\\
&-\frac{240}{m}(m-i)(i+1)+120(m-i+1)
\end{aligned}
$$

注意到

$$
\frac{32}{m^4}(m-i-3)(i+4)(i+3)(i+2)(i+1)>\frac{32}{m^5}(m-i-4)(i+5)(i+4)(i+3)(i+2)(i+1)
$$

所以只需证

$$
\begin{aligned}
A:=&\frac{48}{m^4}(m-i-3)(i+4)(i+3)(i+2)(i+1)-\frac{160}{m^3}(m-i-2)(i+3)(i+2)(i+1)\\
&+\frac{240}{m^2}(m-i-1)(i+2)(i+1)-\frac{240}{m}(m-i)(i+1)+120(m-i+1)>0
\end{aligned}
$$

令 $Y=\frac{i+5}{m}\in(0.5,1]$ 以及 $\epsilon=\frac{1}{m}$, 则

$$
\begin{aligned}
\frac{A}{8m}=&6(1-y+2\epsilon)(y-\epsilon)(y-2\epsilon)(y-3\epsilon)(y-4\epsilon)-20(1-y+3\epsilon)(y-2\epsilon)(y-3\epsilon)(y-4\epsilon)\\
&+30(1-y+4\epsilon)(y-3\epsilon)(y-4\epsilon)-30(1-y+5\epsilon)(y-4\epsilon)+15(1-y+6\epsilon)\\
=&288\epsilon^5+(1584-744y)\epsilon^4+\left(720y^2-2340y+1920\right)\epsilon^3\\
&+\left(960+1270y^2-330y^3-1720y\right)\epsilon^2+\left(210+72y^4-480y+510y^2-300y^3\right)\epsilon\\
&+15-45y-50y^3+60y^2+26y^4-6y^5
\end{aligned}
$$

容易验证, 当 $y\in(0.5,1]$ 时上式中 ϵ 的所有系数非负, 进而 $A>0$, 故断言 2 得证.
至此, 只剩下 $m-4\le i\le m-2$ 的情形.
事实上, 当 $i=m-4$ 时,

$$
\begin{aligned}
\frac{6}{i!}P^{(m-4)}\left(-\frac{2}{m}\right)&\ge 30-\frac{48(m-3)}{m}+\frac{36}{m^2}(m-2)(m-3)-\frac{16}{m^3}(m-1)(m-2)(m-3)\\
&\ge 30-\frac{48(m-3)}{m}+\frac{20}{m^2}(m-2)(m-3)\\
&=\frac{2m^2+44m+120}{m^2}>0.
\end{aligned}
$$

类似地, 对于 $i=m-3,m-2$ 的情形也可类似处理.
综上, 命题得证.
编者按: 以上证明过程根据李皓昭老师提供的证明整理而成. 以下是李老师对此题的点评:
本题来源于凯勒-爱因斯坦度量的相关问题. 复几何中一个重要的问题是凯勒-爱因斯坦度量的存在性, 该存在性与一系列泛函的性质密切相关. 我们需要寻找一个条件, 使得这一系列泛函在某个函数空间中有下界. 研究这个问题其中一步就是此题.

皇榜 · 征解

1 思考题

1. 设 $F(x_1,x_2,x_3,x_4)\in\mathbb{C}[x_1,x_2,x_3,x_4]$ 是一个 4 次齐次不可约多项式, 记

$$J=(F_{x_1},F_{x_2},F_{x_3},F_{x_4})$$

为偏导数 $F_{x_1},F_{x_2},F_{x_3},F_{x_4}$ 生成的理想, J 中 d 次齐次多项式集合记作 J_d, $\mathbb{C}[x_1,x_2,x_3,x_4]$ 中 d 次齐次多项式集合记作 V_d. 假设方程组

$$F=F_{x_1}=F_{x_2}=F_{x_3}=F_{x_4}=0$$

没有非平凡解 (除了 0 点无解, 用代数几何语言是 F 在 $\mathbb{P}^3$ 中定义的超曲面是光滑的).
(1) 证明：方程组 $F_{x_1}=F_{x_2}=F_{x_3}=F_{x_4}=0$ 没有非平凡解；
(2) 计算 $\dim V_d$；
(3) 由 Hilbert 零点定理可知, 对充分大的 n, 有 $V_n=J_n$. 请找一个 n(不依赖于 F) 使得对所有 $d>n$, 有 $V_d=J_d$；
(4) 若存在 $G,H\in V_4$ 满足 $G^2,H^2\in J_8$, $G\cdot H\notin J_8$. 猜想: 存在线性变换 T 与二元多项式 U,V, 满足 $[x_1,x_2,x_3,x_4]=[y_1,y_2,y_3,y_4]T$, 且 $F(x_1,x_2,x_3,x_4)=U(y_1,y_2)+V(y_3,y_4)$.

第 1 题由张磊老师供题, 其中 (4) 源于左康老师的猜想, 目前仍未被解决.

2. 考虑 n 阶随机矩阵 $M_n=(e_{ij})_{n\times n}$, 其中 $\mathbb{P}(e_{ij}=1)=\mathbb{P}(e_{ij}=-1)=1/2$, 且所有分量独立. 本题的目的是证明一个关于估计 $\det M_n$ 的猜想.
(1) 证明：$\mathbb{E}[(\det M_n)^2]=n!$；
(2) 证明：对于任意的函数 $f(n)$, 若 $\lim\limits_{n\to\infty}f(n)=\infty$, 则 $\lim\limits_{n\to\infty}\mathbb{P}(\,|\det M_n|\le f(n)\sqrt{n!}\,)=1$;
(3) 设 X 是 M_n 的第 1 列, W 是 M_n 的第 $2,\dots,d+1$ 列张成的线性子空间, 其中 $1\le d\le n-4$, n 充分大. 记 $d(X,W)$ 表示 n 维 Euclid 空间中向量 X 对应的点到子空间的 W 的距离. 证明: $\forall t>0$,

$$\mathbb{P}(\,|d(X,W)-\sqrt{n-d}|\ge t+1\,)\le 4\exp(-t^2/16)\,;$$

(4) 证明：$\lim\limits_{n\to\infty}\mathbb{P}(\,|\det M_n|\ge g(n)\sqrt{n!}\,)=1$, 其中 $g(n)=\exp(-29\sqrt{n\log n})$;
(5) 证明：$|\det M_n|=n^{n(1/2-o(1))}$ a.s. 成立, 其中 $o(1)$ 表示 $n\to\infty$ 时的无穷小量.

第 2 题由刘党政老师供题.

3. 设 $A_0,A_1,\dots,A_m$ 是 $\mathbb{C}$ 上 $m+1$ 个 n 阶方阵 $(m\ge 2)$, 满足

$$A_0A_i-A_iA_0=A_{i+1},\ A_0A_m=A_mA_0,\ A_jA_k=A_kA_j$$

对所有的 $1\le i\le m-1, 1\le j,k\le m$ 都成立. 若 $A_m\ne O$, 求 n 的最小值 (写成关于 m 的多项式).

4. $M_n(\mathbb{C})$ 为所有 n 阶复方阵构成的线性空间.
(1) 设 M 是 $M_n(\mathbb{C})$ 的子空间, 定义: $N=\{A\in M_n(\mathbb{C})\mid AB-BA\in M,\ \forall B\in M\}$. 若 $A\in N$, 且 $\mathrm{tr}AB=0$, $\forall B\in N$. 证明: A 是幂零方阵.
(2) 设 $M_1\subset M_2$ 是 $M_n(\mathbb{C})$ 的两个子空间, 定义: $N=\{A\in M_n(\mathbb{C})\mid AB-BA\in M_1,\ \forall B\in M_2\}$. 若 $A\in N$, 且 $\mathrm{tr}AB=0$, $\forall B\in N$. 证明: A 是幂零方阵.

第 3, 4 题由陈洪佳老师供题.

5. 设 $\Omega \subset \mathbb{R}^2$ 为一个区域且 $u(x,t) \in L^1_{loc}\big(\Omega \times (0,1) \setminus \Gamma\big)$ 为以下方程的一个非负解

$$\frac{\partial u}{\partial t} = \Delta u, \quad \forall (x,t) \in \big(\Omega \times (0,1)\big) \setminus \Gamma,$$

其中 $\Gamma = \{(\xi(t), t) | t \in (0,1)\}$. 且 $\xi : [0,1] \to \Omega$ 是一条 α-Holder 连续曲线.

(1) 若 $\alpha \in (\frac{1}{2}, 1)$. 证明: $u \in L^1_{loc}\big(\Omega \times (0,1)\big)$.

(2) 若 $\alpha \in (0, \frac{1}{2}]$, (1) 的结论还成立吗?

(3) 设 $\Gamma = \{(\xi_1(t), t) | t \in (0,1)\} \cup \{(\xi_2(t), t) | t \in (0,1)\}$, 其中 $\xi_i : [0,1] \to \Omega (i = 1, 2)$ 为 α-Holder 连续曲线且 $\alpha \in (\frac{1}{2}, 1)$, (1) 的结论还成立吗?

本题为上期皇榜第 3 题, 由李皓昭老师供题.

2 投稿方式

请同学们将思考题的解答发送到数院学生会的官方邮箱 mathsu01@ustc.edu.cn. 具体投稿要求，参见杂志最后一页的《征稿启事》。

趣味数学·找词游戏

从下图中找出 17 个微分流形课程中的关键词.

l	a	s	r	e	v	s	n	a	r	t	d
a	a	f	p	u	o	r	g	g	e	i	l
p	f	i	m	o	r	s	e	i	s	e	e
i	d	b	t	a	u	o	m	t	a	f	i
c	r	e	m	n	n	m	r	n	o	e	f
n	u	r	m	i	e	i	o	g	u	m	l
i	o	g	o	r	b	r	f	l	w	b	a
r	h	n	s	u	h	m	e	o	o	e	s
p	t	i	t	h	a	a	r	f	l	d	q
o	o	i	o	i	n	d	m	d	f	d	u
n	o	o	k	g	o	a	n	a	i	i	e
n	m	o	e	b	i	u	s	i	o	n	d
i	s	u	s	u	b	g	o	i	n	g	i

l	a	s	r	e	v	s	n	a	r	t	d
a	a	f	p	u	o	r	g	g	e	i	l
p	f	i	m	o	r	s	e	i	s	e	e
i	d	b	t	a	u	o	m	t	a	f	i
c	r	e	m	n	n	m	r	n	o	e	f
n	u	r	m	i	e	i	o	g	u	m	l
i	o	g	o	r	b	r	f	l	w	b	a
r	h	n	s	u	h	m	e	o	o	e	s
p	t	i	t	h	a	a	r	f	l	d	q
o	o	i	o	i	n	d	m	d	f	d	u
n	o	o	k	g	o	a	n	a	i	i	e
n	m	o	e	b	i	u	s	i	o	n	d
i	s	u	s	u	b	g	o	i	n	g	i

提示: 所有词都来自王作勤老师微分流形课程目录

http://staff.ustc.edu.cn/~wangzuoq/Courses/16F-Manifolds/index.html.

答案二维码位于文章 *What is a motive?* 末尾, 或关注蜗壳的书院微信公众号获取答案.

趣味数学由周潇翔供稿. 周潇翔, 2016 级少年班学院, PB. 现就读于德国波恩大学, 兴趣方向为代数几何. 邮箱: xx352229@mail.ustc.edu.cn.

致谢

在科大数院的各位老师、校友、同学们的共同努力下，我们在复刊《蛙鸣》第 64 期之后得以继续发行第 65 期《蛙鸣》。在此，编辑部全体成员非常感谢各位投稿人的踊跃来稿，感谢第八届数院学生会的同学参与排版设计和宣传工作。

编辑部在此感谢各位审稿人：姚一晨、范城玮、郑伟豪、付杰、邵锋、陈耀斌、黄一轩、杨威、李树、陈子聪、吴汶政、陈凡、宋秋阳、田珺昊、单逸、伍文超、洪放、范惟。感谢他们在学习和科研的百忙之中抽出时间完成了谨慎、专业的审稿工作。感谢郑伟豪同学协助联络《皇榜》栏目事宜。

编辑部在此感谢中国科学技术大学数学科学学院对第 65 期《蛙鸣》给予的全方位支持。编辑部还特别致谢马杰老师、王作勤老师、郑芳老师对本期《蛙鸣》的编撰、筹备工作的悉心关怀与指导。感谢李皓昭老师、张磊老师、刘党政老师、陈洪佳老师为《皇榜》栏目供题与点评。感谢单墫老师、胡森老师、左康老师、黄文老师、王毅老师、王莉老师、林开亮老师对创办本期《蛙鸣》的热情关切与鼓励。

最后，感谢各位读者阅读本期《蛙鸣》，期待与各位下期再见！

第 65 期《蛙鸣》编辑部

2022 年 6 月 6 日

第 66 期《蛙鸣》征稿启事

现在，我们正式开始为下一期《蛙鸣》征稿！

一、蛙鸣的创刊宗旨

《蛙鸣》是中科大数院的学生杂志。1981 年 6 月 20 日，首期《蛙鸣》由 78 级数学系的同学们自写、自编、自刻、自印而成。四十年来，《蛙鸣》一直是一个完全由学生主导，共同探讨、自由交流数学的开放平台，让同学们可以互相交流彼此的思想和发现，哪怕这些想法并不成熟。所以，我们欢迎各位科大的校友和同学们踊跃投稿！也欢迎外校师生投稿，以增进交流！

二、蛙鸣的栏目设置

目前，《蛙鸣》主要设置如下几个栏目。

（1）**初阳** 初起之阳，朝气蓬勃。"初阳"栏目主要收录大一、大二年级同学的投稿。

（2）**星辰** 我们的征途是星辰大海。"星辰"栏目主要收录人物采访或传记，讲座、报告、座谈实录，数学家的建议，以及数学史、数学科普等稿件。

（3）**蛙声一片** 蛙鸣者，其形虽小，其声也宏。"蛙声一片"栏目收录其它数学类稿件，主要包括小论文、综述报告、定理推广、知识应用、研究前沿讨论等。

（4）**笔墨诗篇** 饱蘸笔墨书诗篇。"笔墨诗篇"栏目主要收录诗篇、随笔等文学类作品。

（5）**皇榜** 大牛贴皇榜，学生揭皇榜。早在 2013-2014 年，科大就有"揭皇榜"擂台赛，由各院系老师出题，难度控制在全校只有极个别的同学能解出来，这正是"满腹经纶无人晓，一揭皇榜天下知"。现在，我们重新开启了"皇榜"栏目，欢迎各路高手接招！

三、投稿方向（包括但不仅限于如下所述）

（1）数学在科学中的应用

（2）小论文、综述报告

（3）研究讨论、前沿介绍

（4）对定理、习题的推广、理解、应用

（5）人物采访或传记

（6）讲座、报告、座谈实录

（7）数学史、数学科普

（8）读书分享、随笔等文学性作品

四、投稿流程与要求

请将稿件发送到数院学生会的官方邮箱: mathsu01@ustc.edu.cn.

（1）邮件标题：蛙鸣_投稿方向_文章标题_作者。邮件正文注明作者信息、联系方式。

（2）推荐中文投稿，但也支持英文投稿。稿件的篇幅应控制在20页以内。英文投稿须语言流畅，无语法错误。

（3）稿件若包含数学公式，则须使用LaTeX排版成PDF（推荐使用官网的模板）。

（4）稿件中请隐去作者信息。

（5）稿件要求写摘要，即以通俗易懂的方式简要介绍文章的内容和主要想法。

（6）稿件需要注明参考文献。若为读书报告，请注明，并进一步提供阅读稿件可能需要的材料出处。

（7）数学类稿件尽量不要写成定理和证明的堆砌、罗列。作者最好用自己的语言叙述，

并写出自己的理解。可通过各种创意增加文章的可读性、趣味性（如插图，支持手绘来稿，由我们绘制电子版）。

（8）稿件被录用后请提供 LaTeX 源代码和参考文献的文件 bib 文件或直接把 bibliography 部分的代码写在 tex 文件里。

五、其它

编委会收到稿件后将回复，并在 2-4 周左右反馈审稿人的初审意见，3 个月内决定是否录用稿件。每篇稿件将至少有 2 位审稿人同时审稿，稿件若被采用则有稿费。

更多信息和 LaTeX 投稿模板下载，请扫描下方左侧的二维码访问数院学生会官方的《蛙鸣》投稿网页 http://staff.ustc.edu.cn/~mathsu01/pu/submit.html.

若有更多问题，请联系数院学生会的官方 QQ 号：蜗壳的书院（QQ: 2061453364）。

《蛙鸣》主页二维码　　　　蜗壳的书院 QQ 二维码

下期预告

截止目前，第 66 期《蛙鸣》已经征得如下稿件：

《蛙鸣记忆》　　　　左康老师

欢迎大家踊跃来稿！

其形虽小
其声也宏
充实基础
奏出强音

谷超豪