RECURSIVE COMPUTATION OF INVARIANT DISTRIBUTION OF FELLER PROCESS: APPLICATIONS AND NUMERICAL EXPERIMENTS

EA MAP

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Ruikai CHEN - Zian CHEN - Tiena SORO

École Polytechnique
IP PARIS

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1
INTRODUCTION

This EA report aims to summarize the work conducted during the initial phase of 3A. Specifically, it focuses on studying the framework of recursive computation using the discretization scheme of Feller processes and proving the convergence of the resulting empirical measures to the invariant distribution. The study is primarily based on the work of Gilles Pagès and Clément Rey in [1], supplemented by additional theoretical and numerical extensions.

Invariant measures are fundamental in characterizing the long-term behavior of Feller process; however, their direct computation is often infeasible due to the complexity of these equations. The core concept involves approaching the Feller process (X_t) with a discretized process $(\overline{X}_{\Gamma_n})$, where a careful discretization is required to ensure both the accuracy and efficiency. Under the proposed framework, the invariant measure of (X_t) can be approximated through the recursive computation of weighted empirical measures of $(\overline{X}_{\Gamma_n})$. By imposing specific assumptions, we obtain the almost sure weak convergence of these measures. This approach significantly simplifies the numerical computation of invariant distributions and reduces the algorithm's space-time complexity.

The report is organized into three sections. In the first section, we outline the framework of the recursive algorithm, including the relevant notations, definitions, assumptions on the scheme and weights, key lemmas, and main theorems.

In the second section, we replicate the recursive algorithm for a new discretization scheme of Ito diffusion, where we conduct deep analysis for the properties of solutions of a stochastic differential equation. This section represents our primary theoretical contribution, where detailed lemmas, calculations, and proofs are provided to ensure a comprehensive understanding of the entire algorithm.

Finally, the third section presents numerical simulations for a simplified one-dimensional case using the Ornstein–Uhlenbeck process as an example. Drawing on [2] and [3], we discuss parameter selection, adjustments, and the verification of the theoretical results established in the second section.

2
GENERAL FRAMEWORK OF RECURSIVE COMPUTATION

2.1 Construction of empirical measure

In this part we present the general framework of recursive computation detailed in [1]. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. We consider a Feller process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{G}, \mathbb{P})$ taking values in a locally compact and separable metric space E. We denote by $(P_t)_{t\geq 0}$ the associated Feller semi-group on $\mathcal{C}_0(E)$, i.e. a collection of positive linear maps from $\mathcal{C}_0(E)$ to itself such that $P_0 f = f$, $P_{t+s}f = P_t P_s f$, $t, s \geq 0$, and $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$. We can then introduce the infinitesimal generator of $(X_t)_{t\geq 0}$ as a linear operator A defined on a subspace $\mathcal{D}(A)$ of $\mathcal{C}_0(E)$, such that

$$\forall f \in \mathcal{D}(A),\ Af = \lim_{t\to 0} \frac{P_t f - f}{t}$$

exists for the $\|\cdot\|_\infty$-norm, and $\mathcal{D}(A)$ is called the domain of A.

We are interested in the computation of the invariant distribution of $(X_t)_{t\geq 0}$, which is characterized by Echeverria–Weiss theorem:

$$\mathcal{V} = \{\nu \in \mathcal{P}(E) : \forall t \geq 0, P_t \nu = \nu\} = \{\nu \in \mathcal{P}(E) : \forall f \in \mathcal{D}(A), \nu(Af) = 0\}.$$

When explicit computation of ν is infeasible or the real process is difficult to simulate, the approximation methods become crucial. We begin by introducing a empirical scheme $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}^*}$ defined on a time grid $\Gamma_n = \sum_{k=1}^n \gamma_k$ with step sizes $(\gamma_n)_{n\in\mathbb{N}^*}$, satisfying:

$$0 < \gamma_n \leq \gamma, \quad \lim_{n\to\infty} \gamma_n = 0, \quad \lim_{n\to\infty} \Gamma_n = +\infty,$$

which is generally a simulatable approximation of $(X_t)_{t\geq 0}$ at a reasonable computational cost. The pseudo-generator $(\tilde{A}_{\gamma_n})_{n\in\mathbb{N}^*}$ of $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}^*}$ is a family of linear operators from $\mathcal{C}_0(E)$ to itself defined by

$$\forall f \in \mathcal{C}_0(E),\ \forall n \in \mathbb{N}^*,\ \tilde{A}_{\gamma_n} f = \frac{\mathcal{Q}_{\gamma_n} f - f}{\gamma_n},$$

where $\mathcal{Q}_{\gamma_n}$ is the transition probability distributions given by $\mathbb{P}(\bar{X}_{\Gamma_n} \in dy | \bar{X}_{\Gamma_{n-1}}) = \mathcal{Q}_{\gamma_n}(\bar{X}_{\Gamma_{n-1}}, dy)$. We can also write explicitly

$$\tilde{A}_{\gamma_n} f(x) = \frac{1}{\gamma_n} \mathbb{E}[f(\bar{X}_{\Gamma_n}) - f(\bar{X}_{\Gamma_{n-1}}) \mid \bar{X}_{\Gamma_{n-1}} = x].$$

Based on $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}^*}$, we can build the empirical measures with a weight sequence $\eta := (\eta_n)_{n\in\mathbb{N}^*}$ such that

$$\forall n \in \mathbb{N}^*,\ \eta_n > 0,\ \lim_n H_n = +\infty \text{ with } H_n = \sum_{k=1}^n \eta_k$$

by defining the random weighted empirical random measures as follows:

$$\nu_n^\eta(\mathrm{d}x) = \frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\bar{X}_{\Gamma_{k-1}}}(\mathrm{d}x),$$

where δ_x is the Dirac delta measure centered at x.

The following part is dedicated to show that under certain conditions, a.s. every weak limiting distribution of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ belongs to $\mathcal{V}$. In particular, when the invariant measure of $(X_t)_{t\geq 0}$ is unique, i.e. $\mathcal{V} = \{\nu\}$, we show that $\lim_n \nu_n^\eta f = \nu f$ $\mathbb{P}$-a.s. for a particular class of continuous test functions f.

2.2 Assumptions

We now present the necessary assumptions in order to prove the convergence of the empirical measures $(\nu_n^\eta)_{n\in\mathbb{N}}$.

Mean-Reverting Recursive Control Let $\bar{\gamma}>0$, let $s\geq 1$, let $\psi,\phi:[v_*,\infty)\to(0,+\infty)$ be Borel functions, and let $\alpha>0$ and $\beta\in\mathbb{R}$. Suppose that there exists a Borel function V, which we call Lyapunov function, if it satisfies

$$\mathcal{L}_V \equiv (V:E\to[v_*,+\infty),\quad v_*>0,\quad \lim_{x\to\infty} V(x)=+\infty) \tag{1}$$

and

$$\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)\equiv\begin{cases}\text{(i)} & \tilde{A}_\gamma\psi\circ V \text{ exists for every } \gamma\in(0,\bar{\gamma}],\\ \text{(ii)} & \forall x\in E,\ \sup_{\gamma\in(0,\bar{\gamma}]}\tilde{A}_\gamma\psi\circ V(x)\leq\frac{\psi\circ V(x)}{V(x)}\big(\beta-\alpha\phi\circ V(x)\big),\\ \text{(iii)} & \liminf_{v\to\infty}\phi(v)>\frac{\beta}{\alpha},\quad\text{and}\quad\lim_{v\to+\infty}\frac{\phi(v)\psi(v)^{1/s}}{v}=+\infty.\end{cases} \tag{2}$$

The function ϕ thus controls the mean-reverting property, we call it strongly mean-reverting property when $\phi=Id$, and weakly mean-reverting property when $\lim_{v\to\infty}\phi(v)/v=0$. We introduce next the sets of functions for which the a.s. convergence holds:

$$C_{\tilde{V}_{\psi,\phi,s}}(E):=\{f\in\mathcal{C}(E):|f(x)|=o(\tilde{V}_{\psi,\phi,s}(x))\},\ \text{where } \tilde{V}_{\psi,\phi,s}(x):=\frac{\phi\circ V(x)\psi\circ V(x)^{1/s}}{V(x)}. \tag{3}$$

Infinitesimal Generator Approximation This assumption control the distance between $(\tilde{A}_\gamma)_{\gamma>0}$ and A. We assume that there exists $\mathcal{D}(A)_0\subset\mathcal{D}(A)$ with $\mathcal{D}(A)_0$ dense in $\mathcal{C}_0(E)$ such that:

$$\mathcal{E}(\tilde{A},A,\mathcal{D}(A)_0)\equiv\forall\gamma\in(0,\bar{\gamma}],\,\forall f\in\mathcal{D}(A)_0,\,\forall x\in E,|\tilde{A}_\gamma f(x)-Af(x)|\leq\Lambda_f(x,\gamma), \tag{4}$$

where $\Lambda_f:E\times\mathbb{R}^+\to\mathbb{R}^+$ can be represented in the following way: Let $(\Omega,\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ be a probability space. Let $g:E\to\mathbb{R}_+^q$, $q\in\mathbb{N}$, be a locally bounded Borel measurable function, and let: $\tilde{\Lambda}_f:(E\times\mathbb{R}^+\times\tilde{\Omega},\,\mathcal{B}(E)\otimes\mathcal{B}(\mathbb{R}^+)\otimes\tilde{\mathcal{G}})\to\mathbb{R}_+^q$ be a measurable function such that:

$$\sup_{i\in\{1,\ldots,q\}}\tilde{\mathbb{E}}\left[\sup_{x\in E}\sup_{\gamma\in(0,\bar{\gamma}]}\tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega})\right]<+\infty,\ \text{and}\ \ \forall x\in E,\,\forall\gamma\in(0,\bar{\gamma}],\Lambda_f(x,\gamma)=\langle g(x),\tilde{\mathbb{E}}[\tilde{\Lambda}_f(x,\gamma,\tilde{\omega})]\rangle_{\mathbb{R}^q}.$$

We assume that for every $i\in\{1,\ldots,q\}$, $\sup_{n\in\mathbb{N}^*}\nu_n^\eta(g_i,\omega)<+\infty$ $\tilde{\mathbb{P}}(d\omega)-$a.s. and that there exists a measurable function $\gamma:(\Omega,\tilde{G})\to((0,\bar{\gamma}],\mathcal{B}((0,\bar{\gamma}]))$ such that one of the following two conditions holds:
(I) $\tilde{\mathbb{P}}(d\tilde{\omega})$-a.s.

$$\begin{cases}\text{(i)} & \forall K\in\mathcal{K}_E,\ \lim_{\gamma\to0}\sup_{x\in K}\tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega})=0,\\ \text{(ii)} & \lim_{x\to\infty}\sup_{\gamma\in(0,\gamma(\tilde{\omega})]}\tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega})=0.\end{cases} \tag{5}$$

(II) $\tilde{\mathbb{P}}(d\tilde{\omega})$-a.s.

$$\lim_{\gamma\to0}\sup_{x\in E}\tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega})g_i(x)=0.$$

Growth Control and Step Weight Assumption We conclude with a hypothesis concerning the steps in the approximation. Let $\rho\in[1,2]$ and $\epsilon_I:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing function. For $F\subset\{f:(E,\mathcal{B}(E))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))\}$ and a Borel function $g:E\to\mathbb{R}^+$, we assume that, for every $n\in\mathbb{N}$,

$$\mathcal{GC}_Q(F,g,\rho,\epsilon_I)\equiv\mathbb{P}\text{-a.s.}\quad\forall f\in F,\quad\mathbb{E}\left[|f(\bar{X}_{\Gamma_{n+1}})-\mathcal{Q}_{\gamma_{n+1}}f(\bar{X}_{\Gamma_n})|^\rho\mid\bar{X}_{\Gamma_n}\right]\leq C_f\epsilon_I(\gamma_{n+1})g(\bar{X}_{\Gamma_n}), \tag{6}$$

where $C_f > 0$ is a finite constant that may depend on f. We suppose that:

$$\mathcal{SW}_{\mathcal{J},\gamma,\eta}(g,\rho,\epsilon_I) \equiv \mathbb{P}\text{-a.s.} \quad \sum_{n=1}^{\infty} \left| \frac{\eta_n}{H_n \gamma_n} \right|^{\rho} \epsilon_I(\gamma_n) g(\bar{X}_{\Gamma_n}) < +\infty, \tag{7}$$

and

$$\mathcal{SW}_{\mathcal{J}\mathcal{J},\gamma,\eta}(F) \equiv \mathbb{P}\text{-a.s.} \quad \forall f \in F, \sum_{n=0}^{\infty} \frac{\left(\frac{\eta_{n+1}}{\gamma_{n+1}} - \frac{\eta_n}{\gamma_n}\right)_+}{H_{n+1}} |f(\bar{X}_{\Gamma_n})| < +\infty. \tag{8}$$

Notice that this last assumption holds as soon as the sequence $(\eta_n/\gamma_n)_{n\in\mathbb{N}^*}$ is non-increasing. Finally we state a lemma which is useful for verifying the last two assumptions (see Lemma 2.3 in [1] for a proof).

Lemma 1. *Let $v^* > 0$, $V : E \to [v^*,\infty)$, and $\psi,\phi : [v^*,\infty) \to \mathbb{R}^+$ be such that $\tilde{A}_{\gamma_n}\psi \circ V$ exists for every $n \in \mathbb{N}^*$. Let $\alpha > 0$, $\beta \in \mathbb{R}$, and $s \geq 1$. Let $(\theta_n)_{n\in\mathbb{N}^*}$ be a non-increasing sequence such that $\sum_{n\geq 1} \theta_n\gamma_n < +\infty$. We assume that $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)$ (see (2)) holds and that $\mathbb{E}[\psi \circ V(\tilde{X}_{\Gamma_{n_0}})] < +\infty$ for every $n_0 \in \mathbb{N}^*$. Then*

$$\sum_{n=1}^{\infty} \theta_n\gamma_n \mathbb{E}[\tilde{V}_{\psi,\phi,1}(X_{\Gamma_{n-1}})] < +\infty,$$

with $\tilde{V}_{\psi,\phi,1}$ defined in (3). In particular, let $\rho \in [1,2]$ and let $\epsilon_I : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. If we also assume that

$$\mathcal{SW}_{\mathcal{J},\gamma,\eta}(\rho,\epsilon_I) \equiv \left(\frac{1}{\gamma_n}\epsilon_I(\gamma_n) \left(\frac{\eta_n}{H_n\gamma_n}\right)^{\rho} \right)_{n\in\mathbb{N}^*} \text{ is non-increasing and } \sum_{n=1}^{\infty} \left(\frac{\eta_n}{H_n\gamma_n}\right)^{\rho} \epsilon_I(\gamma_n) < +\infty, \tag{9}$$

then we have $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(\tilde{V}_{\psi,\phi,1},\rho,\epsilon_I)$ (see (7)). Finally, if

$$\mathcal{SW}_{\mathcal{J}\mathcal{J},\gamma,\eta} \equiv \left(\left(\frac{\eta_{n+1}}{\gamma_{n+1}} - \frac{\eta_n}{\gamma_n}\right)^+ \frac{1}{\gamma_n \times H_n} \right)_{n\in\mathbb{N}^*} \text{ is non-increasing and } \sum_{n=1}^{\infty} \left(\frac{\eta_{n+1}}{\gamma_{n+1}} - \frac{\eta_n}{\gamma_n}\right)^+ \frac{1}{H_n} < +\infty, \tag{10}$$

then we have $\mathcal{SW}_{\mathcal{J}\mathcal{J}},\gamma,\eta(\tilde{V}_{\psi,\phi,1})$ (see (8)).

2.3 Convergence of empirical measure

In this section, we state the main theorems of recursive computation (see Theorem 2.3 and Theorem 2.4 in [1] for a proof). First, we show a tightness property which ensures the existence of a weak limiting distribution for $(\nu_n^\eta)_{n\in\mathbb{N}}$. Then, in a second step, we show that every limiting distributions identified with an invariant distribution of the Feller process $(X_t)_t \geq 0$.

Theorem 1 (Almost sure tightness)**.** *Let $s \geq 1$, $\rho \in [1,2]$, $v^* > 0$, and $V : E \to [v^*,\infty)$, $g : E \to \mathbb{R}^+$, $\psi : [v^*,\infty) \to \mathbb{R}^+$, $\epsilon_I : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function.*
***(A)** Assume that $\tilde{A}_{\gamma_n}(\psi \circ V)^{1/s}$ exists for every $n \in \mathbb{N}^*$, and $\mathcal{GC}_Q((\psi \circ V)^{1/s}, g, \rho, \epsilon_I)$ (see (6)), $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(g,\rho,\epsilon_I)$ (see (7)), and $\mathcal{SW}_{\mathcal{J}\mathcal{J},\gamma,\eta}((\psi \circ V)^{1/s})$ (see (8)) hold. Then*

$$\mathbb{P}\text{-}a.s. \sup_{n\in\mathbb{N}^*} -\frac{1}{H_n}\sum_{k=1}^{n} \eta_k \tilde{A}_{\gamma_k}(\psi \circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}}) < +\infty. \tag{11}$$

***(B)** Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Let $\phi : [v^*,\infty) \to \mathbb{R}_+^*$ be a continuous function such that $C_\phi := \sup_{v\in[v^*,\infty)} \phi(v)/v < +\infty$. Assume that (11), $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)$ (see (2)), $\mathcal{L}_V$ (see (1)) hold. Then*

$$\mathbb{P}\text{-}a.s. \sup_{n\in\mathbb{N}^*} \nu_n^\eta(\tilde{V}_{\psi,\phi,s}) < +\infty,$$

with $\tilde{V}_{\psi,\phi,s}$ defined in (3). Therefore, $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is $\mathbb{P}$-a.s. tight.

Theorem 2 (Identification of the limit)**.** *Let* $\rho \in [1,2]$.
(A) *Let* $\mathcal{D}(A)_0 \subset \mathcal{D}(A)$ *with* $\mathcal{D}(A)_0$ *dense in* $\mathcal{C}_0(E)$. *Assume that* $\tilde{A}_{\gamma_n} f$ *exists for every* $f \in D(A)_0$ *and every* $n \in \mathbb{N}^*$. *Also assume that there exists a Borel function* $g : E \to \mathbb{R}^+$ *and an increasing function* $\epsilon_I : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\mathcal{GC}_Q(\mathcal{D}(A)_0, g, \rho, \epsilon_I)$ *(see (6)), and* $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(g, \rho, \epsilon_I)$ *(see (7)), hold and that*

$$\lim_{n \to +\infty} \frac{1}{H_n} \sum_{k=1}^{n} \left| \frac{\eta_{k+1}}{\gamma_{k+1}} - \frac{\eta_k}{\gamma_k} \right| = 0. \tag{12}$$

Then

$$\mathbb{P}\text{-}a.s.\ \forall f \in \mathcal{D}(A)_0, \lim_{n \to +\infty} \frac{1}{H_n} \sum_{k=1}^{n} \eta_k \tilde{A}_{\gamma_k} f(X_{\Gamma_{k-1}}) = 0. \tag{13}$$

(B) *Assume that (13) and* $\mathcal{E}(\tilde{A}, A, \mathcal{D}(A)_0)$ *(see (4)) hold. Then*

$$\mathbb{P}\text{-}a.s.\ \forall f \in \mathcal{D}(A)_0, \lim_{n \to +\infty} \nu_n^\eta(Af) = 0.$$

It follows that, $\mathbb{P}$*-a.s., every weak limiting distribution* ν_∞^η *belongs to* $\mathcal{V}$. *Finally, if the hypotheses from Theorem 1* ***(B)*** *hold and* $\mathcal{V} = \{\nu\}$, *then*

$$\mathbb{P}\text{-}a.s.\ \forall f \in \tilde{C}_{\tilde{V}_\psi,\phi,s}(E), \lim_{n \to +\infty} \nu_n^\eta(f) = \nu(f).$$

with $\tilde{C}_{V\psi,\phi,s}(E)$ *defined in (3).*

3
APPLICATION TO THE DISCRETIZATION SCHEME OF ITO DIFFUSION

This part is mainly inspired by Section 3.1 of Pagès Gilles, and Clément Rey's work [4], we will treat a new case of the discretization scheme of Ito diffusion with the framework proposed in the previous part. We propose an approach based on polynomial test functions under weakly mean-reverting assumptions. In the first section, we introduce the problem and state a main theorem. Then in this section, we verify all the necessary assumptions and finally give a proof.

3.1 Main result

We consider a d-dimensional Brownian motion $(W_t)_{t\geq 0}$. We are interested in the solution of the d-dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are locally bounded functions. The infinitesimal generator of this process is given by

$$Af(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}Tr(\sigma\sigma^*(x)\nabla^2 f(x)) \tag{14}$$

and its domain $\mathcal{D}(A)$ contains $\mathcal{D}(A)_0 = \mathcal{C}^3_K(\mathbb{R}^d)$. Notice that $\mathcal{D}(A)_0$ is dense in $\mathcal{C}^0(\mathbb{R}^d)$. Assume a Lipschitz condition holds for b and σ:

$$\exists L,\ \forall x, y \in \mathbb{R}^d,\ \|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\|_F \leq L\|x - y\|, \tag{15}$$

where $\|\cdot\|$ is the Euclidean norm, $\|\cdot\|_F$ is the Frobenius norm defined by

$$\|\sigma\|_F := \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{d}|\sigma_{ij}|^2} = \mathrm{Tr}(\sigma^*\sigma)^{\frac{1}{2}},\ \forall \sigma \in \mathbb{R}^{d\times d}.$$

If we pick $y = 0$, the Lipschitz condition implies the following linear growth condition:

$$\exists L',\ \forall x \in \mathbb{R}^d,\ \|b(x)\| + \|\sigma(x)\|_F \leq L'(1 + \|x\|). \tag{16}$$

The Lipschitz condition (15) ensures in fact the existence and uniqueness of the solution $(X_t)_{t\geq 0}$, and it is a Feller process with Feller semi-group $P_t f : x \mapsto \mathbb{E}^x[f(X_t)]$ (see Corollary 19.27 in [5]). We introduce its discretization scheme defined by $(X_{\Gamma_n})_{n\in\mathbb{N}}$, where the times grid $\Gamma_n = \sum_{k=1}^{n}\gamma_k, n \in \mathbb{N}$ satisfies

$$\forall n \in \mathbb{N}^*,\ 0 < \gamma_n \leq \bar{\gamma} := \sup_{n\in\mathbb{N}}\gamma_n < \infty,\ \lim_n \gamma_n = 0,\ \lim_n \Gamma_n = \infty. \tag{17}$$

The pseudo-generator of the scheme $(\tilde{A}_{\gamma_n})_{n\in\mathbb{N}^*}$ is defined by

$$\forall f \in \mathcal{C}_0(\mathbb{R}^d),\ \tilde{A}_{\gamma_n}f(x) = \frac{1}{\gamma_n}\mathbb{E}[f(X_{\Gamma_n}) - f(X_{\Gamma_{n-1}}) \mid X_{\Gamma_{n-1}} = x].$$

Let $v_* > 0$ and let $\phi : [v_*, \infty) \to \mathbb{R}^+$ be a continuous function. Let $p > 0$ and define $\psi_p(y) = y^p$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$, assume that

$$\liminf_{y\to\infty} \phi(y) \geq \frac{\beta}{\alpha}, \; C_\phi := \sup_{y\geq v_*} \frac{\phi(y)}{y} < \infty. \tag{18}$$

We assume that the Lyapunov function $V : \mathbb{R}^d \to [v_*, \infty)$, satisfies $\mathcal{L}_V$:

$$\mathcal{L}_V \equiv (V : E \to [v_*, +\infty), \lim_{x\to\infty} V(x) = +\infty), \tag{19}$$

and is essentially quadratic in the sense

$$\|\nabla V\|^2 \leq C_V V, \; \|D^2 V\|_\infty < +\infty. \tag{20}$$

We define

$$\forall x \in \mathbb{R}^d, \; \lambda_\psi(x) := \lambda_{D^2V(x)+\nabla V(x)^{\otimes 2}\psi''\circ V(x)\psi'\circ V(x)^{-1}}. \tag{21}$$

When $\psi(y) = \psi_p(y) = y^p$, we will also use the notation λ_p instead of λ_ψ. Assume the mean-reverting property of V:

$$\mathcal{R}_p(\alpha, \beta, \phi, V) \equiv \forall x \in \mathbb{R}^d, \langle \nabla V(x), b(x)\rangle + \frac{1}{2}\chi_p(x) \leq \beta - \alpha\phi \circ V(x), \tag{22}$$

$$\text{with } \chi_p(x) = \begin{cases} \|\lambda_1\|_\infty \|\sigma(x)\|_F^2 & \text{if } p \leq 1, \\ \|\lambda_p\|_\infty 2^{(2p-3)_+}\|\sigma(x)\|_F^2 & \text{if } p > 1. \end{cases} \tag{23}$$

And assume that

$$\mathcal{B}(\phi) \equiv \forall x \in \mathbb{R}^d, \; \|\nabla V(x)\| \left(\|b(x)\|^2 + \|\sigma(x)\|_F^2\right)^{\frac{1}{2}} + \|b(x)\|^2 + \|\sigma(x)\|_F^2 \leq C\phi \circ V(x). \tag{24}$$

Remark 1. *When $\phi(y) = y^a$ for $a \in [\frac{1}{2}, 1]$, assumption $\mathcal{B}(\phi)$ (see (24)) can be implied by*

$$\forall x \in \mathbb{R}^d, \; \|b(x)\|^2 + \|\sigma(x)\|_F^2 \leq C' V(x)^{2a-1}. \tag{25}$$

In fact, since $\|\nabla V(x)\| \leq C_V^{\frac{1}{2}} V(x)^{\frac{1}{2}}$, we have

$$\|\nabla V(x)\|(\|b(x)\|^2 + \|\sigma(x)\|_F^2)^{\frac{1}{2}} + \|b(x)\|^2 + \|\sigma(x)\|_F^2 \leq (C_V C')^{\frac{1}{2}} V(x)^a + C'V(x)^{2a-1} \leq CV(x)^a.$$

Theorem 3. *Let $p \geq 1$, $a \in (0,1]$, $s \geq 1$, $\rho \in [1,2]$, $\psi_p(y) = y^p$, $\phi(y) = y^a$ and $\epsilon_I(\gamma) = \gamma^{\frac{\rho}{2}}$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Assume that Lipschitz condition (see (15)), (17), (18), $\mathcal{L}_V$ (see (19)), (20), $\mathcal{R}_p(\alpha, \beta, \phi, V)$ (see (22)), $\mathcal{B}(\phi)$ (see (24)), $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(\rho, \epsilon_I)$ (see (9)), $\mathcal{SW}_{\mathcal{JJ},\gamma,\eta}(V^{p/s})$ (see (8)) and (12) are satisfied, and that $ap\rho \leq p + a - 1$. If $\frac{p}{s} + a - 1 > 0$, then $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is $\mathbb{P} - a.s.$ tight and*

$$\mathbb{P} - a.s. \quad \sup_{n\in\mathbb{N}^*} \nu_n^\eta(V^{p/s+a-1}) < +\infty. \tag{26}$$

Moreover, assume that there is some $\epsilon > 0$, such that $g_\sigma \leq CV^{p/s+a-1}$, with $g_\sigma = (\|b\|^2 + \|\sigma\|_F^2)^{1+\epsilon}$. Then every weak limiting distribution ν of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is an invariant distribution of $(X_t)_{t\geq 0}$, and when ν is unique, we have

$$\mathbb{P} - a.s. \quad \forall f \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d), \; \lim_{n\to\infty} \nu_n^\eta(f) = \nu(f),$$

with $\mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$ defined in (3).

Remark 2.
1. *When $\frac{p}{s} \leq p + a - 1$, assumption $\mathcal{SW}_{\mathcal{JJ},\gamma,\eta}(V^{p/s})$ (see (8)) can be replaced by $\mathcal{SW}_{\mathcal{JJ},\gamma,\eta}$ (see (10)) by Lemma 1.*
2. *When $\frac{p}{s} > 1$, there is some $\epsilon > 0$, such that $g_\sigma \leq CV^{p/s+a-1}$, with $g_\sigma = (\|b\|^2 + \|\sigma\|_F^2)^{1+\epsilon}$ by assumption $\mathcal{B}(\phi)$ (see (18)).*

3.2 Recursive control

Proposition 1. *Under the assumptions of (15), (17), (18), and assume that (19), (20), (22), (24) are satisfied. Then, for every $\tilde{\alpha} \in (0, \alpha)$, there exists $n_0 \in \mathbb{N}^*$, such that*

$$\forall n \geq n_0,\ \forall x \in \mathbb{R}^d,\ \tilde{A}_{\gamma_n} \psi_p \circ V(x) \leq \frac{\psi_p \circ V(x)}{V(x)} p(\beta - \tilde{\alpha}\phi \circ V(x)). \tag{27}$$

Before getting into the main proof, we first introduce some lemmas which are curial in the proof.

Lemma 2. *Let $l \in \mathbb{N}^*$, we have the following inequality:*

$$\forall \alpha > 0,\ \forall u_i \in \mathbb{R}^d,\ i = 1, \cdots, l,\ \left|\sum_{i=1}^{l} u_i\right|^\alpha \leq l^{(\alpha-1)_+} \sum_{i=1}^{l} |u_i|^\alpha. \tag{28}$$

Lemma 3 (Burkholder inequality)**.** *For all $p \geq 2$, there exists $C_p > 0$ s.t.*

$$\mathbb{E}\left[\sup_{s \leq T} \left\| \int_0^s \sigma(X_u) dW_u \right\|^p\right] \leq C_p \mathbb{E}\left[\left(\int_0^T \|\sigma(X_u)\|_F^2 du\right)^{\frac{p}{2}}\right] \tag{29}$$

Proof. Since all the matrix norms are equivalent, we obtain this result by a standard Burkholer inequality (see Theorem 18.17 in [5]). □

Lemma 4. *With the notions above and assume $\gamma_{n+1} \leq 1$, we have*

$$\begin{aligned}\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}] &= \mathbb{E}\left[\left\| \int_{\Gamma_n}^{\Gamma_{n+1}} b(X_s) ds \right\|^2 + \int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds \,\middle|\, X_{\Gamma_n}\right] \\ &\leq \mathbb{E}\left[\int_{\Gamma_n}^{\Gamma_{n+1}} \gamma_{n+1} \|b(X_s)\|^2 + \|\sigma(X_s)\|_F^2 ds \,\middle|\, X_{\Gamma_n}\right].\end{aligned} \tag{30}$$

In particular,

$$\begin{aligned}&\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds \mid X_{\Gamma_n}] = \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}[\|X_s - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}] ds \\ &\leq \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}[\int_{\Gamma_n}^{s} \|b(X_u)\|^2 + \|\sigma(X_u)\|_F^2 du \mid X_{\Gamma_n}] ds \leq \gamma_{n+1} \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|b(X_s)\|^2 + \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}].\end{aligned} \tag{31}$$

Proof. Since the solution of a stochastic differential equation is a homogeneous Markov process (see Section 2 in [6]), we have

$$\mathbb{E}[\Psi(X_{\bullet+s}) \mid \mathscr{F}_s] = \mathbb{E}[\Psi(X_{\bullet+s}) \mid X_s] = \mathbb{E}^{X_s}[\Psi(X_\bullet)], \tag{32}$$

for all bounded measurable functionals Ψ of the sample paths (see Remark 6.3 in [5]). Hence

$$\begin{aligned}\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}] &= \mathbb{E}^{X_{\Gamma_n}}[\|X_{\gamma_{n+1}} - X_0\|^2] \\ &= \mathbb{E}^{X_{\Gamma_n}}\left[\left\| \int_0^{\gamma_{n+1}} b(X_s) ds \right\|^2\right] + \mathbb{E}^{X_{\Gamma_n}}\left[\left\| \int_0^{\gamma_{n+1}} \sigma(X_s) dW_s \right\|^2\right].\end{aligned} \tag{33}$$

The second term is treated by multivariate Ito isometry (see Appendix D, Lemma 18 in [7]):

$$\mathbb{E}^{X_{\Gamma_n}}\left[\left\| \int_0^{\gamma_{n+1}} \sigma(X_s) dW_s \right\|^2\right] = \mathbb{E}^{X_{\Gamma_n}}\left[\int_0^{\gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\right] = \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}]. \tag{34}$$

And we conclude by using the Markov property to the first term in (33). □

Lemma 5. *Suppose that $p \geq 1$, C_{2p} defined as in Lemma 3, let $0 < \delta < 1$ s.t. $1 - 3^{p-1}2^{2p}L^2C_{2p}\delta^p > 0$, let $K > 0$ s.t.*

$$(1 - 3^{p-1}2^{2p}L^2C_{2p}\delta^p)K \geq 3^{p-1} + 3^{p-1}L^p 2^{2p+\frac{1}{2}}C_{2p}^{\frac{1}{2}}\delta^{\frac{p}{2}}K^{\frac{1}{2}}, \tag{35}$$

Suppose $n_0 \in \mathbb{N}$ s.t. $\lambda_{n+1} \leq \delta, \forall n \geq n_0$, then for every $n \geq n_0$, we have

$$\mathbb{E}\Bigg[\Big(\int_{\Gamma_n}^{\Gamma_{n+1}} \|b(X_s)\|^2 ds\Big)^p + \Big(\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\Big)^p \;\Bigg|\; X_{\Gamma_n}\Bigg] \leq K\gamma_{n+1}^p(\|b(X_{\Gamma_n})\|^{2p} + \|\sigma(X_{\Gamma_n})\|_F^{2p}). \tag{36}$$

Proof. Using the fact that (X_t) is a Markov process as mentioned in Lemma 4, (36) is equivalent to

$$\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|b(X_s)\|^2 ds\Big)^p + \Big(\int_0^{\gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\Big)^p\Bigg] \leq K\gamma_{n+1}^p(\|b(X_{\Gamma_n})\|^{2p} + \|\sigma(X_{\Gamma_n})\|_F^{2p}). \tag{37}$$

For simplicity we note

$$P := \mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|b(X_s)\|^2 ds\Big)^p + \Big(\int_0^{\gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\Big)^p\Bigg]. \tag{38}$$

$$Q := \gamma_{n+1}^p(\|b(X_{\Gamma_n})\|^{2p} + \|\sigma(X_{\Gamma_n})\|_F^{2p}). \tag{39}$$

We have

$$\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\Big)^p\Bigg] = \mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} (\|\sigma(X_s)\|_F^2 - \|\sigma(X_{\Gamma_n})\|_F^2) ds + \gamma_{n+1}\|\sigma(X_{\Gamma_n})\|_F^2\Big)^p\Bigg] \tag{40}$$

Since σ is Lipschitz,

$$\begin{aligned}\int_0^{\gamma_{n+1}} (\|\sigma(X_s)\|_F^2 - \|\sigma(X_{\Gamma_n})\|_F^2) ds &= \int_0^{\gamma_{n+1}} \langle 2\sigma(X_{\Gamma_n}), \sigma(X_s) - \sigma(X_{\Gamma_n})\rangle_F + \|\sigma(X_s) - \sigma(X_{\Gamma_n})\|_F^2 ds \\ &\leq 2L\|\sigma(X_{\Gamma_n})\|_F \int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\| ds + L^2 \int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds\end{aligned} \tag{41}$$

By using Lemma 2 and (41), we obtain from (40) that

$$\begin{aligned}\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\Big)^p\Bigg] \leq 3^{p-1}\Bigg((2L)^p\|\sigma(X_{\Gamma_n})\|_F^p \mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\| ds\Big)^p\Bigg] \\ + L^2\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds\Big)^p\Bigg] + \gamma_{n+1}^p\|\sigma(X_{\Gamma_n})\|_F^{2p}\Bigg)\end{aligned} \tag{42}$$

By appling Cauchy's inequality and Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\| ds\Big)^p\Bigg] &\leq \mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\gamma_{n+1}^{\frac{1}{2}}(\int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds))^{\frac{1}{2}}\Big)^p\Bigg] \\ &\leq \gamma_{n+1}^{\frac{p}{2}}\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds\Big)^p\Bigg]^{\frac{1}{2}}.\end{aligned} \tag{43}$$

We compute the last term in (43):

$$\begin{aligned}&\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds\Big)^p\Bigg] = \mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \Big\|\int_0^s b(X_u)du + \int_0^s \sigma(X_u)dW_u\Big\|^2 ds\Big)^p\Bigg] \\ &\leq \mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \Big(2s\int_0^s \|b(X_u)\|^2 du + 2\Big\|\int_0^s \sigma(X_u)dW_u\Big\|^2\Big) ds\Big)^p\Bigg] \\ &\leq \mathbb{E}^{X_{\Gamma_n}}\Bigg[\gamma_{n+1}^{p-1}\int_0^{\gamma_{n+1}} \Big(2s\int_0^s \|b(X_u)\|^2 du + 2\Big\|\int_0^s \sigma(X_u)dW_u\Big\|^2\Big)^p ds\Bigg] \quad \text{(Jensen's inequality)} \\ &\leq \gamma_{n+1}^{2p}2^{2p-1}\mathbb{E}^{X_{\Gamma_n}}\Bigg[\Big(\int_0^{\gamma_{n+1}} \|b(X_u)\|^2 du\Big)^p\Bigg] + \gamma_{n+1}^p 2^{2p-1}\mathbb{E}^{X_{\Gamma_n}}\Bigg[\sup_{s\leq\gamma_{n+1}}\Big\|\int_0^s \sigma(X_u)dW_u\Big\|^{2p}\Bigg] \\ &\leq \gamma_{n+1}^p 2^{2p-1}C_{2p}P \quad \text{(Burkholder inequality, see Lemma 3)}\end{aligned} \tag{44}$$

From (42), (43) and (44) we have

$$\mathbb{E}^{X_{\Gamma_n}}\left[\left(\int_0^{\gamma_{n+1}}\|\sigma(X_s)\|_F^2 ds\right)^p\right] \leq 3^{p-1}(2L)^p 2^{p-\frac{1}{2}} C_{2p}^{\frac{1}{2}} \gamma_{n+1}^{\frac{p}{2}} Q^{\frac{1}{2}} P^{\frac{1}{2}} + 3^{p-1} L^2 2^{2p-1} C_{2p} \gamma_{n+1}^p P + 3^{p-1}\gamma_{n+1}^p \|\sigma(X_{\Gamma_n})\|_F^2. \quad (45)$$

Similarly, by replacing the Frobenius norm of $\sigma(X)$ in (45) by the Euclidean norm of $b(X)$ and adding two inequalities, we get

$$P \leq 3^{p-1} L^p 2^{2p+\frac{1}{2}} C_{2p}^{\frac{1}{2}} \gamma_{n+1}^{\frac{p}{2}} Q^{\frac{1}{2}} P^{\frac{1}{2}} + 3^{p-1} L^2 2^{2p} C_{2p} \gamma_{n+1}^p P + 3^{p-1} Q. \quad (46)$$

Since $\gamma_{n+1} \leq \delta$, from (35) we get

$$(1 - 3^{p-1} 2^{2p} L^2 C_{2p} \gamma_{n+1}^p) K \geq 3^{p-1} + 3^{p-1} L^p 2^{2p+\frac{1}{2}} C_{2p}^{\frac{1}{2}} \gamma_{n+1}^{\frac{p}{2}} K^{\frac{1}{2}},$$

and with (46), we easily see that $P \leq KQ$. □

Corollary 1. *Suppose that $p > 0$, let δ, K, n_0 be defined as in Lemma 5, then for all $n \geq n_0$, there exists $M_p > 0$ which depends on p s.t.*

$$\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^{2p} \mid X_{\Gamma_n}] \leq M_p \gamma_{n+1}^p \left(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2\right)^p. \quad (47)$$

Proof. Case $p \geq 1$. By applying Lemma 2, we have

$$\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^{2p} \mid X_{\Gamma_n}] \leq 2^{2p-1}\mathbb{E}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} b(X_s)ds\right\|^{2p} \,\middle|\, X_{\Gamma_n}\right] + 2^{2p-1}\mathbb{E}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \sigma(X_s)dW_s\right\|^{2p} \,\middle|\, X_{\Gamma_n}\right].$$

The first term can be controlled using Cauchy's inequality:

$$\mathbb{E}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} b(X_s)ds\right\|^{2p} \,\middle|\, X_{\Gamma_n}\right] \leq \gamma_{n+1}^p \mathbb{E}\left[\left(\int_{\Gamma_n}^{\Gamma_{n+1}} \|b(X_s)\|^2 ds\right)^p \,\middle|\, X_{\Gamma_n}\right].$$

And we can treat the second term with Burkholder inequality (see Lemma 3):

$$\mathbb{E}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \sigma(X_s)dW_s\right\|^{2p} \,\middle|\, X_{\Gamma_n}\right] = \mathbb{E}^{X_{\Gamma_n}}\left[\left\|\int_0^{\gamma_{n+1}} \sigma(X_s)dW_s\right\|^{2p}\right] \leq C_{2p}\mathbb{E}^{X_{\Gamma_n}}\left[\left(\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\right)^p\right] = C_{2p}\mathbb{E}\left[\left(\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\right)^p \,\middle|\, X_{\Gamma_n}\right].$$

Then (47) is a direct consequence of Lemma 5.
Case $p < 1$. Since $x \mapsto x^p$ is concave, we can use Jensen's inequality to conclude that

$$E[(\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^2)^p \mid X_{\Gamma_n}] \leq E[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}]^p \leq M_1^p \gamma_{n+1}^p \left(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2\right)^p.$$

□

Proof of Proposition 1. By definition of $\tilde{A}_{\gamma_n}$, (27) is equivalent to

$$\mathbb{E}[\psi \circ V(X_{\Gamma_{n+1}}) - \psi \circ V(X_{\Gamma_n}) \mid X_{\Gamma_n}] \leq \frac{\gamma_{n+1}\psi \circ V(X_{\Gamma_n})}{V(X_{\Gamma_n})}(\beta - \alpha\phi \circ V(X_{\Gamma_n})). \quad (48)$$

Case $p \geq 1$. From Taylor's formula and the definition of λ_p (see (21)) we have

$$\begin{aligned}\psi_p \circ V(X_{\Gamma_{n+1}}) &= \psi_p \circ V(X_{\Gamma_n}) + \langle X_{\Gamma_{n+1}} - X_{\Gamma_n}, \nabla V(X_{\Gamma_n})\rangle \psi_p' \circ V(X_{\Gamma_n}) \\ &\quad + \frac{1}{2}(D^2V(\Upsilon_{n+1})\psi_p' \circ V(\Upsilon_{n+1}) + \nabla V(\Upsilon_{n+1})^{\otimes 2}\psi_p'' \circ V(\Upsilon_{n+1}))(X_{\Gamma_{n+1}} - X_{\Gamma_n})^{\otimes 2} \\ &\leq \psi_p \circ V(X_{\Gamma_n}) + \langle X_{\Gamma_{n+1}} - X_{\Gamma_n}, \nabla V(X_{\Gamma_n})\rangle \psi_p' \circ V(X_{\Gamma_n}) \\ &\quad + \frac{1}{2}\lambda_p(\Upsilon_{n+1})\psi_p' \circ V(\Upsilon_{n+1})|X_{\Gamma_{n+1}} - X_{\Gamma_n}|^2,\end{aligned} \tag{49}$$

with $\Upsilon_{n+1} \in (X_{\Gamma_n}, X_{\Gamma_{n+1}})$. First we deduce from (20) that $\sup_{x\in\mathbb{R}^d} \lambda_p(x) < \infty$. And we have

$$\begin{aligned}&\mathbb{E}[\langle X_{\Gamma_{n+1}} - X_{\Gamma_n}, \nabla V(X_{\Gamma_n})\rangle \mid X_{\Gamma_n}] - \gamma_{n+1}\langle b(X_{\Gamma_n}), \nabla V(X_{\Gamma_n})\rangle \\ &= \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle b(X_s) - b(X_{\Gamma_n}), \nabla V(X_{\Gamma_n})\rangle ds \mid X_{\Gamma_n}] \leq L\|\nabla V(X_{\Gamma_n})\|\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|X_s - X_{\Gamma_n}\| ds \mid X_{\Gamma_n}] \\ &\leq L\gamma_{n+1}^{\frac{1}{2}}\|\nabla V(X_{\Gamma_n})\|\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^2 ds \mid X_{\Gamma_n}]^{\frac{1}{2}},\end{aligned} \tag{50}$$

where we used (43) with $p = 1$ for the last inequality. Using Lemma 4 (see (31)) and Lemma 5 (see (36) with $p = 1$), we deduce from (50) that

$$\begin{aligned}&\mathbb{E}[\langle X_{\Gamma_{n+1}} - X_{\Gamma_n}, \nabla V(X_{\Gamma_n})\rangle \mid X_{\Gamma_n}] - \gamma_{n+1}\langle b(X_{\Gamma_n}), \nabla V(X_{\Gamma_n})\rangle \\ &\qquad \leq L\gamma_{n+1}^{\frac{3}{2}}\|\nabla V(X_{\Gamma_n})\|\mathbb{E}[\frac{1}{\gamma_{n+1}}\int_{\Gamma_n}^{\Gamma_{n+1}} \|b(X_s)\|^2 + \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}]^{\frac{1}{2}} \\ &\qquad \leq LK^{\frac{1}{2}}\gamma_{n+1}^{\frac{3}{2}}\|\nabla V(X_{\Gamma_n})\|(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{1}{2}},\end{aligned} \tag{51}$$

for all $n \geq n_0$ with n_0, K defined as in Lemma 5. From Lemma 4 (see (30)) we have

$$\begin{aligned}\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}] &\leq \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \gamma_{n+1}\|b(X_s)\|^2 + \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}] \\ &\leq \gamma_{n+1}^2\mathbb{E}[\frac{1}{\gamma_{n+1}}\int_{\Gamma_n}^{\Gamma_{n+1}} \|b(X_s)\|^2 + \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}] + \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}]\end{aligned} \tag{52}$$

For the last term of (52), we use (45) and Lemma 5 (see (36)) with $p = 1$:

$$\begin{aligned}&\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds \mid X_{\Gamma_n}] - \gamma_{n+1}\|\sigma(X_{\Gamma_n})\|_F^2 \leq 2^{\frac{3}{2}}LC_2^{\frac{1}{2}}\gamma_{n+1}^{\frac{1}{2}}Q^{\frac{1}{2}}P^{\frac{1}{2}} + 2L^2C_2\gamma_{n+1}P \\ &\leq (2^{\frac{3}{2}}LC_2^{\frac{1}{2}}K^{\frac{1}{2}}\gamma_{n+1}^{\frac{1}{2}} + 2L^2C_2\gamma_{n+1}K)Q = \gamma_{n+1}^{\frac{3}{2}}C(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2),\end{aligned} \tag{53}$$

with $C = 2^{\frac{3}{2}}LC_2^{\frac{1}{2}}K^{\frac{1}{2}} + (2L^2C_2 + 1)\gamma_{n+1}^{\frac{1}{2}}K$, where we adapt the notations P and Q defined in (38) and (39) for a more concise writing. Together with (52) and (53), we obtain

$$\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}] \leq \gamma_{n+1}\|\sigma(X_{\Gamma_n})\|_F^2 + \gamma_{n+1}^{\frac{3}{2}}C(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2), \tag{54}$$

Assume first that $p = 1$. From (49), (51) and (54) we have

$$\begin{aligned}\mathbb{E}[V(X_{\Gamma_{n+1}}) - V(X_{\Gamma_n}) \mid X_{\Gamma_n}] &\leq \gamma_{n+1}\left(\langle b(X_{\Gamma_n}), \nabla V(X_{\Gamma_n})\rangle + \frac{1}{2}\|\lambda_1\|_\infty\|\sigma(X_{\Gamma_n})\|_F^2\right) \\ &\quad + \gamma_{n+1}^{\frac{3}{2}}\left(LK^{\frac{1}{2}}\|\nabla V(X_{\Gamma_n})\|(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{1}{2}} + \frac{1}{2}\|\lambda_1\|_\infty C(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)\right).\end{aligned}$$

By using $\mathcal{B}(\phi)$ (see (24)), for every $\tilde{\alpha} \in (0,\alpha)$, there exists $n_0(\tilde{\alpha}) > n_0$ such that for every $n \geq n_0(\tilde{\alpha})$,

$$\gamma_{n+1}^{\frac{3}{2}}\big(LK^{\frac{1}{2}}\|\nabla V(X_{\Gamma_n})\|(\|b(X_{\Gamma_n})\|^2+\|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{1}{2}} + \frac{1}{2}\|\lambda_1\|_\infty C(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)\big) \\ \leq \gamma_{n+1}(\alpha-\tilde{\alpha})\phi \circ V(X_{\Gamma_n}).$$

From assumption $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (22)), we conclude that

$$\tilde{A}_{\gamma_n}\psi \circ V(x) \leq \beta - \tilde{\alpha}\phi \circ V(x).$$

Assume now that $p > 1$. Since $\|\nabla V\|^2 \leq C_V V$, $\sqrt{V}$ is Lipschitz. We use Lemma 2 to obtain that

$$V^{p-1}(\Upsilon_{n+1}) \leq (\sqrt{V}(X_{\Gamma_n}) + [\sqrt{V}]_1\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|)^{2p-2} \leq 2^{(2p-3)_+}V^{p-1}(X_{\Gamma_n}) + [\sqrt{V}]_1^{2p-2}\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^{2p-2}. \tag{55}$$

Then by using (51), (54), (55) and Corollary 1 (see (47)), we derive from (49) that

$$\mathbb{E}[V^p(X_{\Gamma_{n+1}}) - V^p(X_{\Gamma_n}) \mid X_{\Gamma_n}] \leq \gamma_{n+1}\langle b(X_{\Gamma_n}), \nabla V(X_{\Gamma_n})\rangle pV^{p-1}(X_{\Gamma_n}) \\ + \frac{1}{2}\gamma_{n+1}\|\lambda_p\|_\infty 2^{(2p-3)_+}pV^{p-1}(X_{\Gamma_n})\|\sigma(X_s)\|_F^2 + R,$$

$$\text{with } R = \gamma_{n+1}^{\frac{3}{2}}LK^{\frac{1}{2}}\|\nabla V(X_{\Gamma_n})\|(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{1}{2}}pV^{p-1}(X_{\Gamma_n}) \\ + \gamma_{n+1}^{\frac{3}{2}}\frac{1}{2}\|\lambda_p\|_\infty 2^{(2p-3)_+}pV^{p-1}(X_{\Gamma_n})C(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2) \\ + \gamma_{n+1}^{p}\frac{p}{2}\|\lambda_p\|_\infty[\sqrt{V}]_1^{2p-2}M\left(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2\right)^p.$$

Notice that $\phi \circ V(x) < C_\phi V(x)$ (see (18)) and by using $\mathcal{B}(\phi)$ (see (24)), for every $\tilde{\alpha} \in (0,\alpha)$, there exists $n_0(\tilde{\alpha}) > n_0$ such that for every $n \geq n_0(\tilde{\alpha})$,

$$R \leq \gamma_{n+1}(\alpha-\tilde{\alpha})pV^{p-1}(X_{\Gamma_n})\phi \circ V(X_{\Gamma_n}),$$

which leads to the recursive control for $p > 1$ from assumption $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (22)).
Case $p < 1$. Since $x \mapsto x^p$ is concave, we have

$$V^p(X_{\Gamma_{n+1}}) - V^p(X_{\Gamma_n}) \leq pV^{p-1}(X_{\Gamma_n})(V(X_{\Gamma_{n+1}}) - V(X_{\Gamma_n})).$$

The recursive control for $p < 1$ follows immediately the case $p = 1$ which we have just proved. □

3.3 Proof of the infinitesimal estimation

Proposition 2. *Assume the Lipschitz condition (15) and that there exists $\epsilon > 0$ s.t.*

$$\sup_{n\in\mathbb{N}^*} \nu_n^\eta\big((1 + \|b\|^2 + \|\sigma\|_F^2)^\epsilon(\|b\|^2 + \|\sigma\|_F^2)\big) < +\infty \tag{56}$$

Then $\mathscr{E}(\tilde{A}, A, \mathcal{D}_0(A))$ (see (4)) holds, i.e. there is $\bar{\gamma}$ such that

$$\forall\gamma \in (0,\bar{\gamma}],\ \forall f \in \mathcal{C}_K^3(\mathbb{R}^d),\ \forall x \in \mathbb{R}^d,\ \|\tilde{A}_\gamma f(x) - Af(x)\| \leq \Lambda_f(x,\gamma),$$

where $\Lambda_f(x,\gamma) = g(x)\tilde{\Lambda}_f(x,\gamma)$ with $g : \mathbb{R}^d \to \mathbb{R}_+$ a locally bounded Borel measurable function. Moreover, g and $\tilde{\Lambda}_f$ satisfy $\sup_{n\in\mathbb{N}_*}\nu_n^\eta(g) < +\infty$ and

$$\begin{cases} (i)\ \forall K \in \mathcal{K}_{\mathbb{R}^d},\ \lim_{\gamma\to 0}\sup_{x\in K}\tilde{\Lambda}_f(x,\gamma) = 0, \\ (ii)\ \lim_{x\to\infty}\sup_{\gamma\in(0,\bar{\gamma}]}\tilde{\Lambda}_f(x,\gamma) = 0. \end{cases} \tag{57}$$

We remark that (57) is in fact a special case of (5).

Proof. For $f \in \mathcal{C}^3_K(\mathbb{R}^d)$, we note d the diameter of $supp(f)$, i.e. $d = \max_{x \in supp(f)} \|x\|$. To treat the pseudo-generator $\tilde{A}_\gamma$, we first apply an Itô formula:

$$\mathbb{E}[f(X_{\Gamma_{n+1}}) - f(X_{\Gamma_n}) \mid X_{\Gamma_n}] = \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle b(X_s), \nabla f(X_s)\rangle ds + \int_{\Gamma_n}^{\Gamma_{n+1}} \frac{1}{2} Tr(\sigma\sigma^*(X_s)\nabla^2 f(X_s))ds \mid X_{\Gamma_n}]. \quad (58)$$

We begin by studying the first term of (58):

$$\begin{aligned} &\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle b(X_s), \nabla f(X_s)\rangle ds \mid X_{\Gamma_n}] - \gamma_{n+1}\langle b(X_{\Gamma_n}), \nabla f(X_{\Gamma_n})\rangle \\ &\qquad = \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle \nabla f(X_s) - \nabla f(X_{\Gamma_n}), b(X_{\Gamma_n})\rangle + \langle \nabla f(X_s), b(X_s) - b(X_{\Gamma_n})\rangle ds \mid X_{\Gamma_n}]. \end{aligned} \quad (59)$$

We observe that, if $\|X_s - X_{\Gamma_n}\| < \|X_{\Gamma_n}\| - d$, which also implies $\|X_{\Gamma_n}\| > d$, we can deduce that X_{Γ_n}, X_s do not belong to $supp(f)$. Thus we can add the indicator function $\mathbb{1}_{\|X_s - X_{\Gamma_n}\| \geq \|X_{\Gamma_n}\| - d}$ in the RHS of (59). Let $\alpha > 0$ such that $\frac{\alpha}{2} < \epsilon$, since $\nabla f \in \mathcal{C}^2_K(\mathbb{R}^d)$, it is $\alpha-$Hölder continuous, and we define $|\nabla f|_{C^{0,\alpha}}$ to be its Hölder coefficient. Then the first term in the RHS of (59) can be controled as:

$$\begin{aligned} &\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle \nabla f(X_s) - \nabla f(X_{\Gamma_n}), b(X_{\Gamma_n})\rangle ds \mid X_{\Gamma_n}] \\ &\leq |\nabla f|_{C^{0,\alpha}} \|b(X_{\Gamma_n})\| \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \|X_s - X_{\Gamma_n}\|^\alpha \mathbb{1}_{\|X_s - X_{\Gamma_n}\| \geq \|X_{\Gamma_n}\| - d} ds \mid X_{\Gamma_n}] \\ &\overset{\text{Hölder}}{\leq} |\nabla f|_{C^{0,\alpha}} \|b(X_{\Gamma_n})\| \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}[\|X_s - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}]^{\frac{\alpha}{2}} \mathbb{E}[\mathbb{1}_{\|X_s - X_{\Gamma_n}\| \geq \|X_{\Gamma_n}\| - d} ds \mid X_{\Gamma_n}]^{\frac{2-\alpha}{2}} ds \\ &\leq \gamma_{n+1} |\nabla f|_{C^{0,\alpha}} \|b(X_{\Gamma_n})\| \sup_{\Gamma_n \leq s \leq \Gamma_{n+1}} \mathbb{E}[\|X_s - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}]^{\frac{\alpha}{2}} \sup_{\Gamma_n \leq s \leq \Gamma_{n+1}} \mathbb{E}[\mathbb{1}_{\|X_s - X_{\Gamma_n}\| \geq \|X_{\Gamma_n}\| - d} \mid X_{\Gamma_n}]^{\frac{2-\alpha}{2}}. \end{aligned}$$

For the last term, we can use Markov's inequality:

$$\mathbb{E}[\mathbb{1}_{\|X_s - X_{\Gamma_n}\| \geq \|X_{\Gamma_n}\| - d} \mid X_{\Gamma_n}] \leq \min\left(\frac{\mathbb{E}[\|X_s - X_{\Gamma_n}\|^2 \mid X_{\Gamma_n}]}{(\|X_{\Gamma_n}\| - d)^2}, 1 \right) \mathbb{1}_{\|X_{\Gamma_n}\| > d} + \mathbb{1}_{\|X_{\Gamma_n}\| \leq d}$$

By applying Corollary 1 we conclude that

$$\frac{1}{\gamma_{n+1}} \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle \nabla f(X_s) - \nabla f(X_{\Gamma_n}), b(X_{\Gamma_n})\rangle ds \mid X_{\Gamma_n}] \leq g_1(X_{\Gamma_n}) \tilde{\Lambda_{f,1}}(X_{\Gamma_n}, \gamma_{n+1}),$$

with

$$g_1(x) = |\nabla f|_{C^{0,\alpha}} \|b(x)\| M_1^{\frac{\alpha}{2}} (\|b(x)\|^2 + \|\sigma(x)\|_F^2)^{\frac{\alpha}{2}} (1 + \|b(x)\|^2 + \|\sigma(x)\|_F^2)^\delta,$$

$$\tilde{\Lambda_{f,1}}(x, \gamma) = \frac{\gamma^{\frac{\alpha}{2}}}{(1 + \|b(x)\|^2 + \|\sigma(x)\|_F^2)^\delta} \left(\min\left(\frac{(M_1\gamma)^{\frac{2-\alpha}{2}} (\|b(x)\|^2 + \|\sigma(x)\|_F^2)^{\frac{2-\alpha}{2}}}{(\|x\| - d)^{2-\alpha}}, 1 \right) \mathbb{1}_{\|x\| > d} + \mathbb{1}_{\|x\| \leq d} \right),$$

where M_1 is defined as in Corollary 1, $\delta > 0$ s.t. $\frac{\alpha}{2} + \delta < \epsilon$. Since b and σ have sub-linear growth (see (16)), it's easy to see that $\tilde{\Lambda_{f,1}}$ satisfies the condition (57). And from (56), we obtain that $\sup_{n \in \mathbb{N}_*} \nu_n^\eta(g_1) < +\infty$. Similarly, we use the fact that b is L-Lipschitz to the second term in the RHS of (59), and we obtain

$$\frac{1}{\gamma_{n+1}} \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} \langle \nabla f(X_s), b(X_s) - b(X_{\Gamma_n})\rangle ds \mid X_{\Gamma_n}] \leq g_2(X_{\Gamma_n}) \tilde{\Lambda_{f,2}}(X_{\Gamma_n}, \gamma_{n+1}),$$

with

$$g_2(x) = L\|\nabla f\|_\infty M_1^{\frac{1}{2}} (\|b(x)\|^2 + \|\sigma(x)\|_F^2)^{\frac{1}{2}} (1 + \|b(x)\|^2 + \|\sigma(x)\|_F^2)^\delta,$$

$$\tilde{\Lambda_{f,2}}(x,\gamma) = \frac{\gamma^{\frac{1}{2}}}{(1+\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\delta}}\left(\min\left(\frac{M_1^{\frac{1}{2}}\gamma^{\frac{1}{2}}(\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\frac{1}{2}}}{\|x\|-d},1\right)\mathbb{1}_{\|x\|>d}+\mathbb{1}_{\|x\|\leq d}\right).$$

For the second term of (58), we write the matrix trace as a Frobenius inner product, and obtain a similar equation as (58):

$$\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}}\frac{1}{2}\langle\sigma\sigma^*(X_s),\nabla^2 f(X_s)\rangle_F ds \mid X_{\Gamma_n}] - \gamma_{n+1}\frac{1}{2}\langle\sigma\sigma^*(X_{\Gamma_n}),\nabla^2 f(X_{\Gamma_n})\rangle_F$$
$$= \frac{1}{2}\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}}\langle\sigma\sigma^*(X_{\Gamma_n}),\nabla^2 f(X_s)-\nabla^2 f(X_{\Gamma_n})\rangle_F + \langle\sigma\sigma^*(X_s)-\sigma\sigma^*(X_{\Gamma_n}),\nabla^2 f(X_s)\rangle_F ds \mid X_{\Gamma_n}].$$

Since σ is L-Lipschitz, and using the fact that $\|\sigma\sigma^*\|_F\leq\|\sigma\|_F^2$, we have

$$\|\sigma\sigma^*(X_s)-\sigma\sigma^*(X_{\Gamma_n})\|_F \leq L^2\|X_s-X_{\Gamma_n}\|^2 + 2L\|\sigma(X_{\Gamma_n})\|_F\|X_s-X_{\Gamma_n}\|.$$

Therefore, from a similar reasoning, by using Cauchy's inequality to the term

$$\mathbb{E}[\|X_s-X_{\Gamma_n}\|^2\mathbb{1}_{\|X_s-X_{\Gamma_n}\|>\|X_{\Gamma_n}\|-d} \mid X_{\Gamma_n}]$$

and Corollary 1 with $p=2$, we obtain

$$\frac{1}{\gamma_{n+1}}\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}}\langle\sigma\sigma^*(X_s)-\sigma\sigma^*(X_{\Gamma_n}),\nabla^2 f(X_s)\rangle_F ds \mid X_{\Gamma_n}] \leq g_3(X_{\Gamma_n})\tilde{\Lambda_{f,3}}(X_{\Gamma_n},\gamma_{n+1}),$$

with

$$g_3(x) = \|\nabla^2 f\|_{F,\infty}(2LM_1^{\frac{1}{2}}+L^2M_2^{\frac{1}{2}}\gamma^{\frac{1}{2}})(\|b(x)\|^2+\|\sigma(x)\|_F^2)(1+\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\delta},$$
$$\tilde{\Lambda_{f,3}}(x,\gamma) = \frac{\gamma^{\frac{1}{2}}}{(1+\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\delta}}\left(\min\left(\frac{M_1^{\frac{1}{2}}\gamma^{\frac{1}{2}}(\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\frac{1}{2}}}{\|x\|-d},1\right)\mathbb{1}_{\|x\|>d}+\mathbb{1}_{\|x\|\leq d}\right),$$

and

$$\frac{1}{\gamma_{n+1}}\mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}}\langle\sigma\sigma^*(X_{\Gamma_n}),\nabla^2 f(X_s)-\nabla^2 f(X_{\Gamma_n})\rangle_F ds \mid X_{\Gamma_n}] \leq g_4(X_{\Gamma_n})\tilde{\Lambda_{f,4}}(X_{\Gamma_n},\gamma_{n+1}),$$

with

$$g_4(x) = |\nabla^2 f|_{C^{0,\alpha}}\|\sigma(x)\|_F^2 M_1^{\frac{\alpha}{2}}(\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\frac{\alpha}{2}}(1+\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\delta},$$
$$\tilde{\Lambda_{f,4}}(x,\gamma) = \frac{\gamma^{\frac{\alpha}{2}}}{(1+\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\delta}}\left(\min\left(\frac{(M_1\gamma)^{\frac{2-\alpha}{2}}(\|b(x)\|^2+\|\sigma(x)\|_F^2)^{\frac{2-\alpha}{2}}}{(\|x\|-d)^{2-\alpha}},1\right)\mathbb{1}_{\|x\|>d}+\mathbb{1}_{\|x\|\leq d}\right).$$

To conclude, we have $\|\tilde{A_\gamma}f(x)-Af(x)\| \leq \sum_{i=1}^4 g_i(x)\tilde{\Lambda_{f,i}}(x,\gamma)$. Since there is some constant C s.t. $g_i \leq C(1+\|b\|^2+\|\sigma\|_F^2)^{\epsilon}(\|b\|^2+\|\sigma\|_F^2)$ and $\tilde{\Lambda_{f,i}}(x,\gamma)$ satisfies (57) for all $i=1,2,3,4$, we have completed the proof. □

3.4 Proof of Growth control and Step Weight assumptions

Proposition 3. *Let $p>0$, $a\in(0,1]$, $s\geq 1$, $\rho\in[1,2]$, $\psi(y)=y^p$ and $\phi(y)=y^a$. We assume (15) and (17). Then for every $n\in\mathbb{N}$, for every $f\in\mathcal{D}(A)_0=\mathcal{C}_K^3(\mathbb{R}^d)$,*

$$\mathbb{E}[|f(X_{\Gamma_{n+1}})-f(X_{\Gamma_n})|^{\rho} \mid X_{\Gamma_n}] \leq C_f\gamma_{n+1}^{\frac{\rho}{2}}(\|b(X_{\Gamma_n})\|^2+\|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{\rho}{2}}. \quad (60)$$

In other words, we have $\mathcal{GC}_Q(\mathcal{D}(A)_0,g_\sigma,\rho,\epsilon_I)$ (see (6)) with $g_\sigma=(\|b\|^2+\|\sigma\|_F^2)^{\frac{\rho}{2}}$ and $\epsilon_I(\gamma)=\gamma^{\frac{\rho}{2}}$. Moreover, if (20) and $\mathcal{B}(\phi)$(see (24)) hold and $ap\rho/s\leq p+a-1$. Then for every $n\in\mathbb{N}$, we have

$$\mathbb{E}[\|V^{p/s}(X_{\gamma_{n+1}})-V^{p/s}(X_{\Gamma_n})\|^{\rho} \mid X_{\Gamma_n}] \leq C\gamma_{n+1}^{\frac{\rho}{2}}V^{p+a-1}(X_{\Gamma_{n+1}}). \quad (61)$$

In other words, we have $\mathcal{GC}_Q(V^{p/s},V^{p+a-1},\rho,\epsilon_I)$ (see (6)) with $\epsilon_I(\gamma)=\gamma^{\frac{\rho}{2}}$.

Proof. By using Corollary 1 with $p = \frac{1}{2}$, we know that there is $n_0 \in \mathbb{N}$ s.t.

$$\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\| \mid X_{\Gamma_n}] \leq \gamma_{n+1}^{\frac{1}{2}} M_{\frac{1}{2}} (\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{1}{2}}.$$

The rest of proof is the same as in Lemma 3.8 in [4]. □

3.5 Proof of Theorem 3

The proof consists of verifying all the assumption introduced in Theorem 1 and Theorem 2.

Step 1. Mean-reverting recursive control. Since (20), $\mathcal{B}(\phi)$ (see (24)) and $\mathcal{R}_p(\alpha, \beta, \phi, V)$ (see (22)) hold, it follows from Proposition 1 that $\mathcal{RC}_{Q,V}(\psi_p, \phi, p\tilde{\alpha}, p\beta, s)$ (see (2)) is satisfied for every $\tilde{\alpha} \in (0, \alpha)$ and every $s \geq 1$ such that $p/s + a - 1 > 0$.

Step 2. Step weight assumption. We have $\mathcal{SW}_{\mathcal{IJ},\gamma,\eta}(V^{p/s})$ (see (8)) from assumption, and we can show that $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(V^{p+a-1}, \rho, \epsilon_I)$ (see (7)) by Lemma 1.

Step 3. Growth control assumption. Now, we prove $\mathcal{GC}_{\mathcal{Q}}(F, V^{p+a-1}, \rho, \epsilon_I)$ (see (6)) for $F = \mathcal{D}(A)_0$ and $F = \{V^{p/s}\}$. We notice that $\frac{\rho}{2} \leq 1$, thus $\frac{a\rho}{2} \leq p + a - 1$. From $\mathcal{B}(\phi)$ (see (18)), we obtain that

$$(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2)^{\frac{\rho}{2}} \leq C^{\frac{\rho}{2}} V^{\frac{a\rho}{2}}(X_{\Gamma_n}) \leq C^{\frac{\rho}{2}} V^{a+p-1}(X_{\Gamma_n}).$$

Then Proposition 3 shows that $\mathcal{GC}_{\mathcal{Q}}(F, V^{p+a-1}, \rho, \epsilon_I)$ (see (6)) for $F = \mathcal{D}(A)_0$ and $F = \{V^{p/s}\}$.

Step 4. Conclusion.

1. The first part of Theorem 3 (see (26)) is a consequence of Theorem 1. We obtain from Steps 2 and 3 that assumptions $\mathcal{GC}_{\mathcal{Q}}(V^{p/s}, V^{p+a-1}, \rho, \epsilon_I)$ (see (6)), $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(V^{p+a-1}, \rho, \epsilon_I)$ (see (7)) and $\mathcal{SW}_{\mathcal{IJ},\gamma,\eta}(V^{p/s})$ (see (8)) hold, which are the hypotheses from Theorem 1 (A) with $g = V^{p+a-1}$.

   Due to Step 1, assumption $\mathcal{RC}_{Q,V}(\psi_p, \phi, p\tilde{\alpha}, p\beta, s)$ (see (2)) is satisfied for every $\tilde{\alpha} \in (0, \alpha)$ and every $s \geq 1$ such that $p/s + a - 1 > 0$. Moreover, since $\mathcal{L}_V$ (see (19)) holds, the hypotheses from Theorem 1 (B) are satisfied. We thus conclude from Theorem 1 that $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is $\mathbb{P}-$ a.s.tight and (26) holds.

2. The second part of Theorem 3 comes from Theorem 2. We obtain from Steps 2 and 3 that assumptions $\mathcal{GC}_{\mathcal{Q}}(\mathcal{D}(A)_0, V^{p+a-1}, \rho, \epsilon_I)$ (see (6)) and $\mathcal{SW}_{\mathcal{J},\gamma,\eta}(V^{p+a-1}, \rho, \epsilon_I)$ (see (7)) hold, which are the hypotheses from Theorem 2 (A) with $g = V^{p+a-1}$.

   Since there is some $\epsilon > 0$, such that $g_\sigma \leq CV^{p/s+a-1}$, with $g_\sigma = (\|b\|^2 + \|\sigma\|_F^2)^{1+\epsilon}$, we have $\sup_{n\in\mathbb{N}^*} \nu_n^\eta(g_\sigma) < +\infty$. It follows from Proposition 2 that $\mathscr{E}(\tilde{A}, A, \mathcal{D}_0(A))$ (see (4)) holds. Then the hypotheses from Theorem 2 (B) hold, which leads to the conclusion.

4
NUMERICAL EXPERIMENTS

In the previous section, we have examined the theoretical perspective. In the following, we will utilize `Python` to conduct numerical experimental simulations to verify the theoretical results.

We first need to choose a suitable process which should have an invariant distribution as time t tends to infinity. By writing our process in the following form

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,$$

we have to pick appropriate functions σ and b. One option is the Ornstein–Uhlenbeck process. Let σ be a positive constant and $b(x, y) = \theta(\mu - y)$ with $\theta > 0$. Then, we may have

$$X_t \sim \mathcal{N}(\mu + (x_0 - \mu)e^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})),$$

where we mention that its invariant distribution is $\mathcal{N}(\mu, \sigma^2/(2\theta))$. Thereafter, for the numerical simulation, we take $\theta = 2$, $\sigma = 2$ and $\mu = 1$. Thus, the invariant distribution becomes $\mathcal{N}(1, 1)$.

To get a basic idea, we visualize the general properties of the Ornstein–Uhlenbeck process by showing the following two figures.

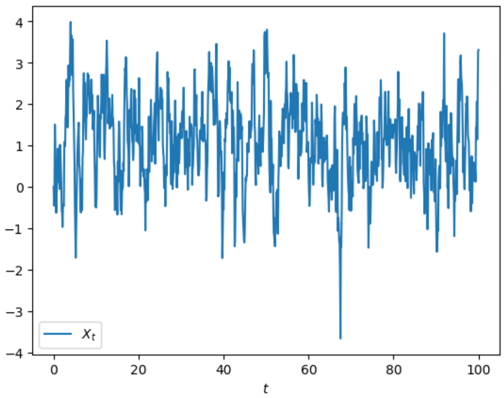

Figure 1: Trajectory of O-U process

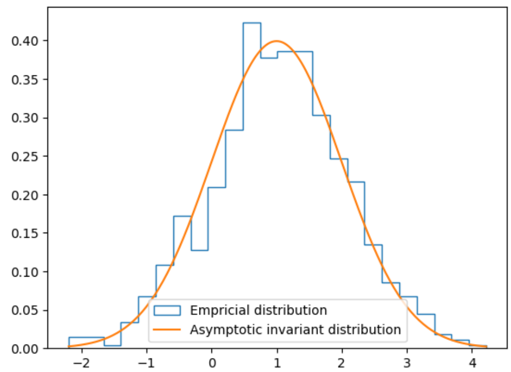

Figure 2: Distribution of O-U process

4.1 Convergence of the Euler Scheme

We may now verify that, given f enough regular, by noting ν the invariant distribution, one can have $\nu_n^\gamma(f) \to \nu(f)$ almost surely, where

$$\nu_n^\gamma(f) := \frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k f(\overline{X_{\Gamma_{k-1}}})$$

with $\Gamma_n = \sum_{k=1}^{n} \gamma_k$ and $\overline{X_t}$ the Euler Scheme, which is the theorem that we have shown.

For the numerical computation of X_t, for general functions b and σ, we use the so-called Euler Scheme. Additionally, in order to consider γ, we define here a stochastic difference equation:

$$\overline{X_{\Gamma_{i+1}}} - \overline{X_{\Gamma_i}} = b(\Gamma_i, \overline{X_{\Gamma_i}})\gamma_{i+1} + \sigma(\Gamma_i, \overline{X_{\Gamma_i}})\sqrt{\gamma_{i+1}}Z_i,$$

where $(Z_i)_{i\geq 0}$ are standard Gaussian.

Finally, consider γ_k in order to satisfy the conditions of the theorem, we take here $\gamma_k = k^{-\alpha}$ with $\alpha \in (0,1)$, such that $\gamma_k \to 0$ and $\Gamma_k \to +\infty$ when $k \to +\infty$. Taking $\alpha = 0.3$, 0.5 and 0.7, we can show the histogram at $n = 10000$, with $f(x) = x$ which gives us the expectation, in the figure 3.

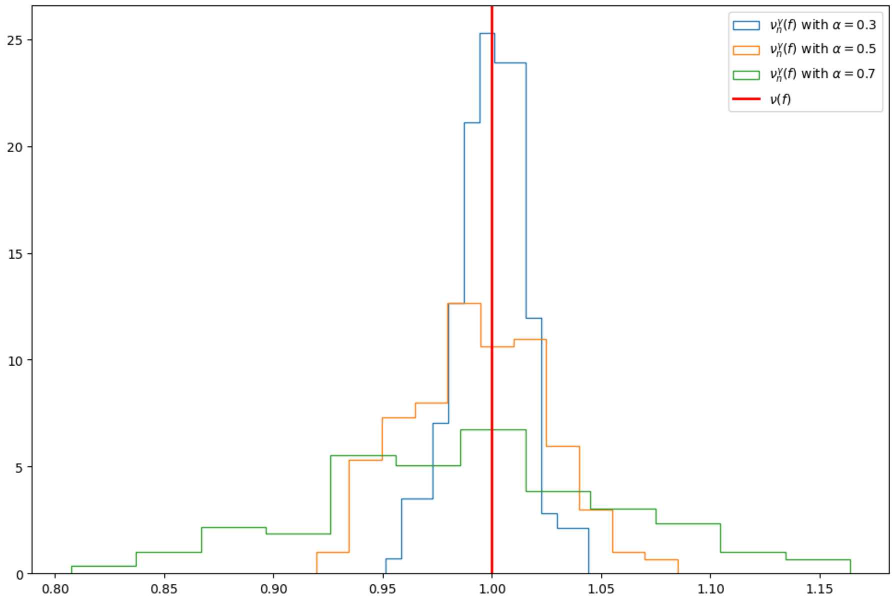

Figure 3: Histogram of $\nu_n^\gamma(f)$ for different α

We may mention that its histogram looks like the normal distribution, because this process, at the time t fix, is a Gaussian. Then, we show its convergence by increasing n in the figure 4.

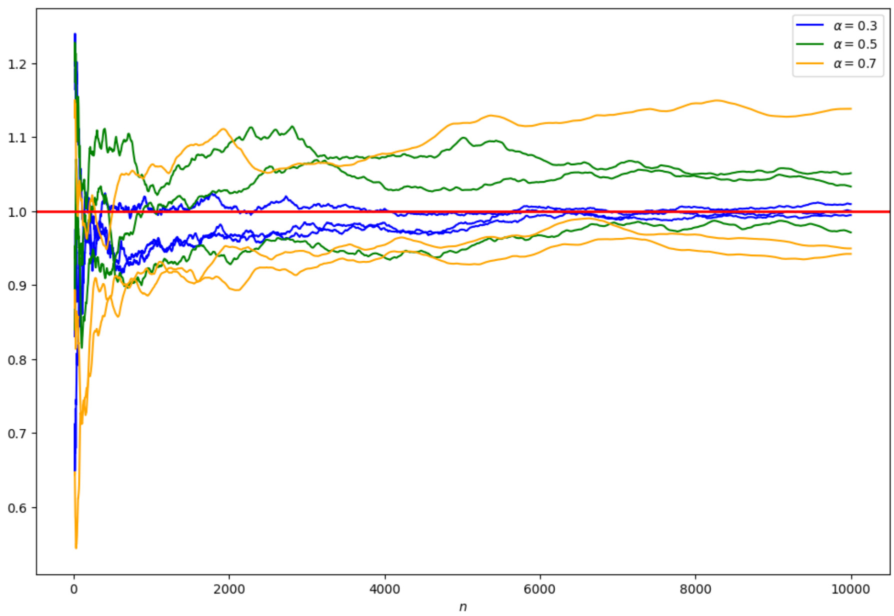

Figure 4: Values of $\nu_n^\gamma(f)$ for different α

With these two figures, we note that $\alpha = 0.3$ seems to be the best case. From the Remark 3.4 of reference

[2], we know that, under certain conditions, the optimum is obtained at $\alpha = 1/3$, which we will discuss in the following subsection.

4.2 Rate of Convergence

When we wish to accelerate the rate of convergence, for specific problems, there are two possible cases to analyse: adjusting γ_k, in particular the exponent α; and for the specific Ornstein–Uhlenbeck process, a better approach than the Euler Scheme can be used.

4.2.1 • Choice of γ_k

Under the assumption that $\gamma_k = k^{-\alpha}$, we may note numerically that although α can take on all values between $(0, 1)$, once α takes on a value close to 0, it will lead to an explosion of constants.

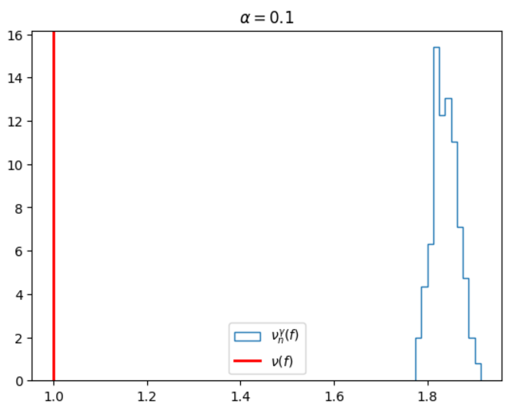

Figure 5: Histogram of $\nu_n^\gamma(f)$ for $\alpha = 0.1$

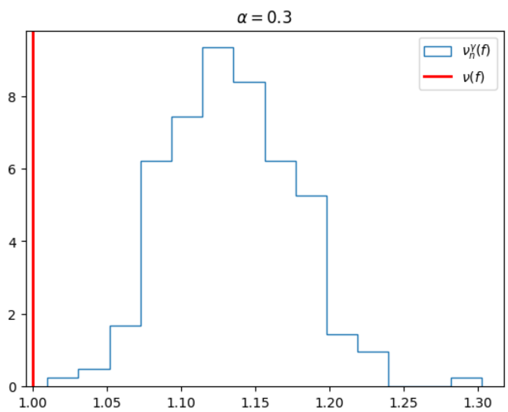

Figure 6: Histogram of $\nu_n^\gamma(f)$ for $\alpha = 0.3$

We used here $f(x) = (x - \mu)^2$ which gives us the variance. There may be two factors taken into account, one is that the value of Γ_n will become smaller when α becomes larger, which will contribute less to X_T which is close to the invariant distribution, for T given and large; on the other hand, γ_k will become larger when α becomes smaller, which will reduce the fineness and cause the Euler Scheme to be different from the real one.

The solution to this problem can actually be very simple, and in order to ensure that γ_k is not too large when α is small, it is sufficient to perform a translation. Specifically, we can define

$$\gamma_k = (k + n_0)^{-\alpha},$$

where n_0 is a fixed constant. Our next step will be about the choice of α.

According to the Theorem 3.2 of reference [2], in case of diffusion process, for f of the form Ag, where A is the infinitesimal generator (see (14)) and g is a function, the optimal α will be 1/3. More precisely, a process is a diffusion process when both b and σ are functions of X_t itself. Taking $g(x) = (x-1)/(1+(x-1)^2)$, we calculate then

$$\begin{aligned} f(x) =& Ag(x) = b(x)g'(x) + \frac{1}{2}\sigma^2(x)g''(x) \\ =& 2(1-x) \cdot \frac{1-(x-1)^2}{(1+(x-1)^2)^2} + 2 \cdot \frac{-4(x-1)}{(1+(x-1)^2)^3} = \frac{2(1-x)(5-(x-1)^4)}{(1+(x-1)^2)^3}. \end{aligned}$$

We mention that $x \mapsto f(x+1)$ is an odd function, hence $\nu(f)$ is exactly 0. We may take here $n = 10000$ and show the normalized absolute errors in heat-map form.

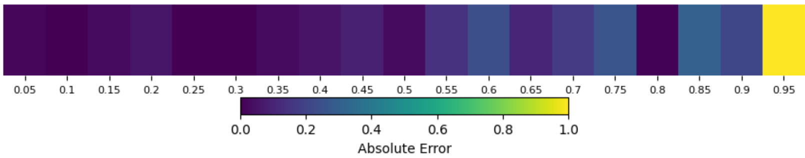

Figure 7: Normalized absolute errors of $\nu_n^{\gamma}(f)$ for different α

This heat-map figure 7 shows that the error is significantly lower around the theoretical optimal choice of α.

4.2.2 • Recursive Formula of the Ornstein–Uhlenbeck Process

The Ornstein–Uhlenbeck process can actually be simulated directly because we have a concrete solution:

$$X_t = \mu + (x_0 - \mu)\exp(-\theta t) + \sigma \int_0^t \exp(-\theta(t-s))\mathrm{d}W_s.$$

While, it may be emphasised that since the γ_k is not homogeneous (it gets smaller as k gets larger), we cannot use the conventional simulation method of dividing the time equally. Obviously, the detail we want to deal with here is the treatment of the integral, so we want to compute the integral numerically in steps of γ_k.

By denoting

$$I_i = \exp(-\Gamma_i)\int_0^{\Gamma_i} \exp(\theta s)\mathrm{d}W_s,$$

we will have

$$\exp(\theta\Gamma_i)I_i - \exp(\theta\Gamma_{i-1})I_{i-1} = \int_{\Gamma_{i-1}}^{\Gamma_i} \exp(\theta s)\mathrm{d}W_s.$$

So, if the integral is replaced by $\exp(\theta\Gamma_{i-1})\sqrt{\gamma_i}Z_i$, we may have the discretised recursive formula for I_i:

$$\overline{I_i} = \exp(-\theta\gamma_i)\overline{I_{i-1}} + \exp(-\theta\gamma_i)\sqrt{\gamma_i}Z_i.$$

We can notice here that Z_i is preceded by an exponent in addition to $\sqrt{\gamma_i}$, which increases a little the speed of convergence somewhat compared to the Euler Scheme, as we will show in the following numerical simulations. We fix here the $f : x \mapsto (x-\mu)^2$ giving the variance.

We also want to verify that $\nu_n^{\gamma}(f)$ will converge to $\nu(f)$ almost surely, and compare to the convergence of the Euler Scheme, where $\overline{X_t}$ is replaced by X_t in the expression of $\nu_n^{\gamma}(f)$. This is proved in the previous section: for diffusion process, there can be convergence for the direct use of the exact X_t when right conditions are satisfied.

As our functions b and σ are extremely simple affine or constant functions, the conditions for regularity are satisfied. It is sufficient to take $V(x) = x^2$ as the Lyapunov function to satisfy the rest of the conditions. Thus, the theoretical proof tells us that there is indeed an almost sure convergence, which we are going to verify by numerical simulation.

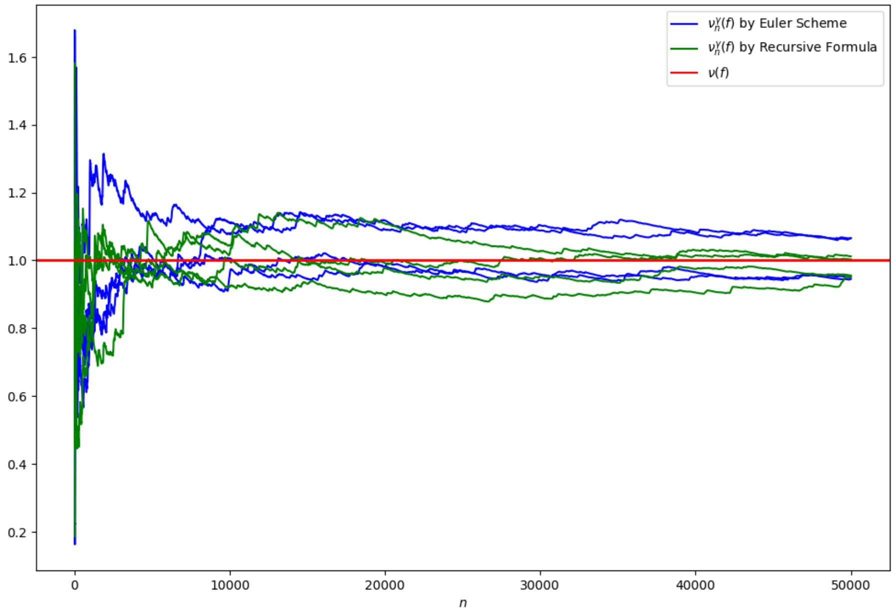

Figure 8: Values of $\nu_n^\gamma(f)$ for different approach

We can notice a slight advantage of using recursive formula over Euler Scheme, most likely due to the factor $\exp(-\theta\gamma_i)$. It is clear that this advantage is not obvious. This is probably because both the Euler Scheme and the recursive formula have the same order, first-order error.

4.3 Related Application

An application of this framework interesting is to prove the almost sure Central Limit Theorem, which we can find as the Theorem 7 of reference [3]. To simplify the conditions, we may consider the one-dimensional case. Let (U_n) be a sequence of independent and identically distributed square-integrable random variables, satisfying $\mathbb{E}[U_1] = 0$ and $\mathrm{Var}(U_1) = 1$. Then, we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{(U_1+\cdots+U_k)/\sqrt{k}} \to \mathcal{N}(0,1)$$

in distribution almost surely.

We now want to verify numerically this theorem. By denoting $\nu = \mathcal{N}(0,1)$ and $\nu_k = \delta_{(U_1+\cdots+U_k)/\sqrt{k}}$, we want to show that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\nu_k(f) \to \nu(f)$$

almost surely, whose form is very similar to what we have concerned if we take $\gamma_k = 1/k$.

We now want to verify numerically this theorem. In the numerical simulation, we may use $U_1 \sim \mathrm{U}(-\sqrt{3},\sqrt{3})$ the uniform distribution, and take $f(x) = x$ for the expectation and $f(x) = x^2$ for the variance.

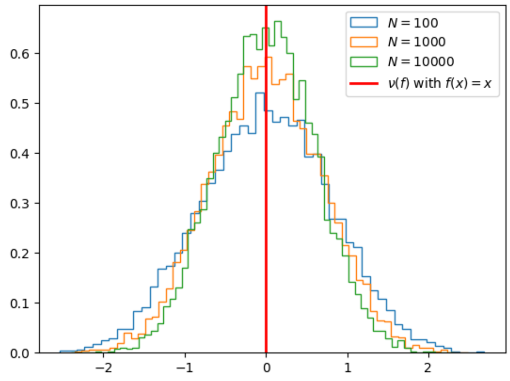

Figure 9: Convergence of expectation

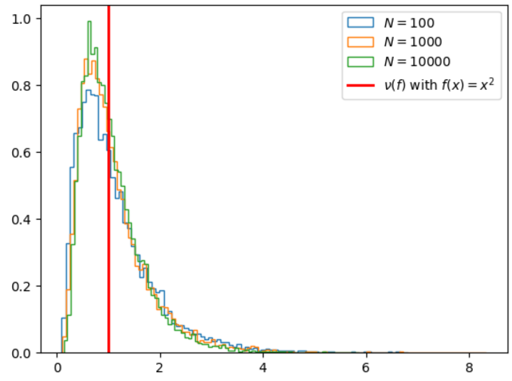

Figure 10: Convergence of variance

We can see that the distribution becomes concentrated as N becomes larger. Since logarithmic functions do not grow as fast as power functions, the rate of its convergence is not very fast even for large N.

4.4 Case of Non-Uniqueness

We may consider a classical example about dual potential well function. We denote

$$b(x) = \begin{cases} 1, & x < -3\pi/2, \\ \sin x, & -3\pi/2 \leq x \leq 3\pi/2, \\ -1, & x > 3\pi/2. \end{cases}$$

One can mention that b is the negative derivative of a potential V shown below.

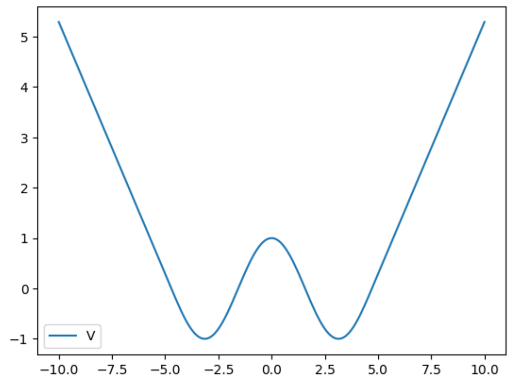

Figure 11: Image of function V

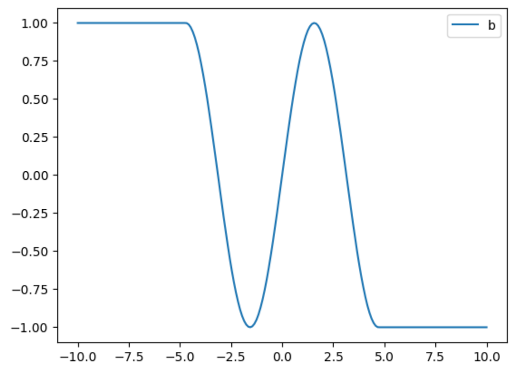

Figure 12: Image of function b

We know that the ordinary differential equation $x' = b(x)$ is initial-value sensitive, converging to π if the initial value is positive, to $-\pi$ if $x_0 < 0$, and $x = 0$ will always remain at the unstable point. Hence, we consider the following stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where σ can be changed. We show histograms by taking $f(x) = x$ and $\alpha = 1/3$.

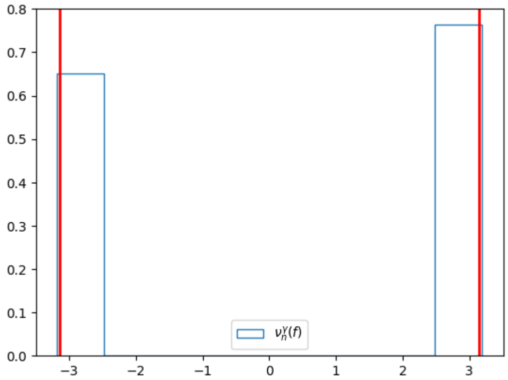

Figure 13: Histogram with $\sigma = 0.5$

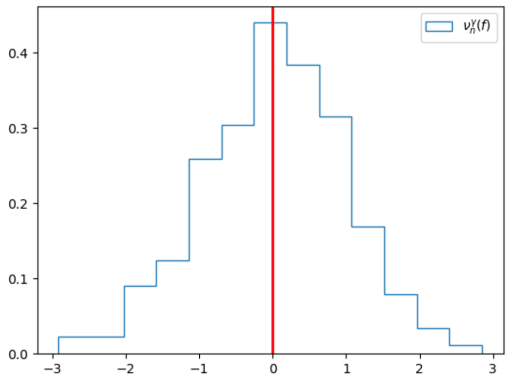

Figure 14: Histogram with $\sigma = 1.5$

One can find that $\nu_n^\gamma(f)$ converges differently while σ changes. In particular, the limit of $\nu_n^\gamma(f)$ is not unique when σ is small. Intuitively, the parameter σ can be thought of as reflecting the step size of a certain random motion, which is difficult to move away from the local minima when the step size is small and easy to move around the center when the step size is large.

5
CONCLUSION

In this article, our primary focus is to comprehend a series of foundational works, apply the key theorems presented to derive new results, and subsequently verify these results through numerical simulations. The central objective of this study is to develop a general methodology for determining the invariant measure of a stochastic differential equation, which enables us to achieve highly accurate approximations with the simulation of a single trajectory, significantly reducing the space-time complexity of the algorithm. A major component of our work also involves applying the proposed method to an alternative scheme, rigorously validating all the assumptions to deepen our understanding of the approach. Furthermore, we have numerically investigated optimization possibilities, such as identifying the optimal exponent to enhance convergence.

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