Recursive Computation of Invariant Distribution of Feller Process: Applications and Numerical experiments

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Sommaire

General Framework

Feller process $(X_t)_t$ $\longrightarrow$ numerical scheme $(\overline{X}_{\Gamma_n})_n$

$$\nu \xleftarrow[\text{convergence}]{\text{a.s. weak}} \nu_n^{\eta}(dx) = \frac{1}{\sum \eta_k} \sum_{k=1}^{n} \eta_k \delta_{\overline{X}_{\Gamma_n}}(dx)$$

invariant measure

empirical invariant measure

Notations

- $(X_t)_t$ is a Feller process with Feller semi-group $(P_t)_t$
- $(\bar{X}_{\Gamma_n})_n$ is a Markov approximation of $(X_t)_t$
- V Lyapunov function
- ψ test function, ϕ control function
- A is the infinitesimal generator : $Af = \lim_{t \to 0} \frac{P_t f - f}{t}$, where f is a continous function
- $\tilde{A}_\gamma$ is the pseudo-generator of $(\bar{X}_{\Gamma_n})$:

$$\tilde{A}_{\gamma_{n+1}} f(x) = \frac{1}{\gamma_{n+1}} \mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_n}) \mid \bar{X}_{\Gamma_n} = x]$$

$\mathcal{D}(A)_0$ set of f where $\tilde{A}_\gamma f$ is well defined.

Main Assumptions

- **Mean-Reverting Recursive Control** :

$$\forall x \in E, \sup_{\gamma \in (0,\bar{\gamma}]} \tilde{A}_\gamma \psi \circ V(x) \leq \frac{\psi \circ V(x)}{V(x)} (\beta - \alpha \phi \circ V(x))$$

- **Infinitesimal Generator Approximation** :

$$\forall \gamma \in (0, \bar{\gamma}], \forall f \in \mathcal{D}(A), \forall x \in E, |\tilde{A}_\gamma f(x) - Af(x)| \leq \Lambda_f(x, \gamma)$$

- **Growth Control** : $\forall f \in F$

$$\mathbb{E}\left[\left| f(\bar{X}_{\Gamma_{n+1}}) - \mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}) | \bar{X}_{\Gamma_n}] \right|^\rho \mid \bar{X}_{\Gamma_n} \right] \leq C_f \epsilon_I(\gamma_{n+1}) g(\bar{X}_{\Gamma_n})$$

- **Step Weight assumptions**

Main results

Theorem (Almost sure tightness, Identification of the limit)

Suppose $s \geq 1$ and that certain assumptions hold, we have

$$\mathbb{P}\text{-}a.s. \sup_{n \in \mathbb{N}^*} \nu_n^\eta \left(\tilde{V}_{\psi,\phi,s} \right) < +\infty,$$

with $\tilde{V}_{\psi,\phi,s} = \frac{\phi \circ V(x) \psi \circ V(x)^{1/s}}{V(x)}$. *Therefore,* $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ *is a.s. tight, and*

$$\mathbb{P}\text{-}a.s. \ \forall f \in \mathcal{D}(A), \lim_{n \to +\infty} \nu_n^\eta (Af) = 0.$$

It follows that, $\mathbb{P}$*-a.s., every weak limiting distribution* ν_∞^η *is a invariant measure. And if the invariant measure* ν *is unique, then for all* f *continuous s.t.* $f = o(\tilde{V}_{\psi,\phi,s})$, *we have a.s.* $\lim \nu_n^\eta(f) = \nu(f)$.

Ito diffusion

We consider the solution of the d-dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$.

Assume a Lipschitz condition holds for b and σ :

$$\exists L,\ \forall x, y \in \mathbb{R}^d,\ \|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\|_F \leq L\|x - y\|,$$

which ensures the existence and uniqueness of the solution $(X_t)_{t\geq 0}$, $(X_t)_{t\geq 0}$ is a Feller process with Feller semi-group

$$P_t f : x \mapsto \mathbb{E}^x[f(X_t)].$$

Notation : $\|\sigma\|_F := Tr(\sigma\sigma^*)^{1/2}$ Frobenius norm.

Discretization scheme of Ito diffusion

When the exact solution $(X_t)_{t\geq 0}$ is known, we can build its empirical invariant measures with a weight sequence $\eta := (\eta_n)_{n\in\mathbb{N}^*}$ and a times grid $\Gamma_n = \sum_{k=1}^{n} \gamma_k$:

$$\nu_n^{\eta}(\mathrm{d}x) = \frac{1}{\sum_{k=1}^{n} \eta_k} \sum_{k=1}^{n} \eta_k \delta_{X_{\Gamma_{k-1}}}(\mathrm{d}x).$$

We will show that, under certain conditions, every weak limiting distribution ν of $(\nu_n^{\eta})_{n\in\mathbb{N}^*}$ is an invariant distribution of (X_t).

Some crucial hypotheses

Let $V : \mathbb{R}^d \to [v_*, \infty)$, $(v_* > 0)$ be a Lyapunov function, $\lim_{x\to\infty} V(x) = +\infty$, and is essentially quadratic in the sense

$$\|\nabla V\|^2 \leq C_V V, \ \|D^2 V\|_\infty < +\infty.$$

Let $\phi : [v_*, \infty) \to \mathbb{R}^+$ be a continuous function. Let $p > 0$ and define $\psi_p(y) = y^p$.

- **Mean-reverting assumptions**

$$\forall x, \ \underbrace{\langle \nabla V(x), b(x)\rangle + \frac{1}{2} C(V,p) Tr(\sigma\sigma^*(x))}_{\approx AV(x)} \leq \beta - \alpha\phi \circ V(x)$$

where $AV(x) := \lim_{t\to 0} \frac{P_t V(x) - V(x)}{t} = \lim_{t\to 0} \frac{1}{t}(\mathbb{E}^x[V(X_t)] - x)$.

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- **Coefficient control**

$$\forall x, \ \|\nabla V(x)\|(\|b(x)\|^2 + \|\sigma(x)\|_F^2)^{\frac{1}{2}} + \|b(x)\|^2 + \|\sigma(x)\|_F^2 \leq C\phi \circ V(x).$$

- **Assumptions on time steps and weights**

Key properties of Ito diffusion

Proposition (Burkholder inequality)

For all $p \geq 2$, there exists $C_p > 0$ s.t.

$$\mathbb{E}\left[\sup_{s\leq T}\left\| \int_0^s \sigma(X_u)dW_u\right\|^p\right] \leq C_p\mathbb{E}\left[\left(\int_0^T \|\sigma(X_u)\|_F^2 du\right)^{\frac{p}{2}}\right]$$

Lemma

Suppose that $p \geq 1$, we have

$$\mathbb{E}\left[\left(\int_{\Gamma_n}^{\Gamma_{n+1}} \|b(X_s)\|^2 ds\right)^p + \left(\int_{\Gamma_n}^{\Gamma_{n+1}} \|\sigma(X_s)\|_F^2 ds\right)^p \,\middle|\, X_{\Gamma_n}\right]$$
$$\leq K\gamma_{n+1}^p(\|b(X_{\Gamma_n})\|^{2p} + \|\sigma(X_{\Gamma_n})\|_F^{2p}).$$

Key properties of Ito diffusion

Corollary

Suppose that $p > 0$, then $\exists n_0, M_p$ s.t. for all $n \geq n_0$,

$$\mathbb{E}[\|X_{\Gamma_{n+1}} - X_{\Gamma_n}\|^{2p} \mid X_{\Gamma_n}] \leq M_p \gamma_{n+1}^p \left(\|b(X_{\Gamma_n})\|^2 + \|\sigma(X_{\Gamma_n})\|_F^2\right)^p .$$

A powerful tool for the control of quantities in the form

$$\mathbb{E}[f(X_{\Gamma_{n+1}}) - f(X_{\Gamma_n}) \mid X_{\Gamma_n}] \text{ and } \mathbb{E}[\int_{\Gamma_n}^{\Gamma_{n+1}} f(X_s)ds \mid X_{\Gamma_n}] - \gamma_{n+1} f(X_{\Gamma_n}).$$

Example (Recursive Control) :

$$\mathbb{E}[\psi \circ V(X_{\Gamma_{n+1}}) - \psi \circ V(X_{\Gamma_n}) \mid X_{\Gamma_n}] \leq \frac{\gamma_{n+1} \psi \circ V(X_{\Gamma_n})}{V(X_{\Gamma_n})} (\beta - \alpha \phi \circ V(X_{\Gamma_n})).$$

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Example (Infinitesimal Estimation)

$$\forall \gamma \in (0,\bar{\gamma}],\ \forall f \in \mathcal{C}_K^3(\mathbb{R}^d),\ \forall x \in \mathbb{R}^d,\ \|\tilde{A}_\gamma f(x)-Af(x)\| \leq \Lambda_f(x,\gamma),$$

with

$$\tilde{A}_{\gamma_n} f(x) = \frac{1}{\gamma_n}\mathbb{E}[f(X_{\Gamma_n}) - f(X_{\Gamma_{n-1}}) \mid X_{\Gamma_{n-1}} = x].$$

$$Af(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2} Tr(\sigma\sigma^*(x)\nabla^2 f(x))$$

Basic Setting

The Ornstein–Uhlenbeck process under our consideration is

$$dX_t = 2(1 - X_t)dt + 2dW_t.$$

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About the time grid and the weight sequence, we take here both $(\gamma_k)_k$ in order to simplify the argument.

Euler Scheme

A general stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

can be discretized as

$$\overline{X_{\Gamma_{i+1}}} - \overline{X_{\Gamma_i}} = b(\Gamma_i, \overline{X_{\Gamma_i}})\gamma_{i+1} + \sigma(\Gamma_i, \overline{X_{\Gamma_i}})\sqrt{\gamma_{i+1}}Z_i,$$

where $(Z_i)_{i\geq 0}$ are standard Gaussian.

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where $(Z_i)_{i \geq 0}$ are standard Gaussian.

Hence, one can have

$$\nu_n^{\gamma}(f) := \frac{1}{\Gamma_n} \sum_{k=1}^{n} \gamma_k f(\overline{X_{\Gamma_{k-1}}}),$$

where we take $\gamma_k = k^{-\alpha}$ and $f(x) = x$ to show the convergence.

Convergnce Figures

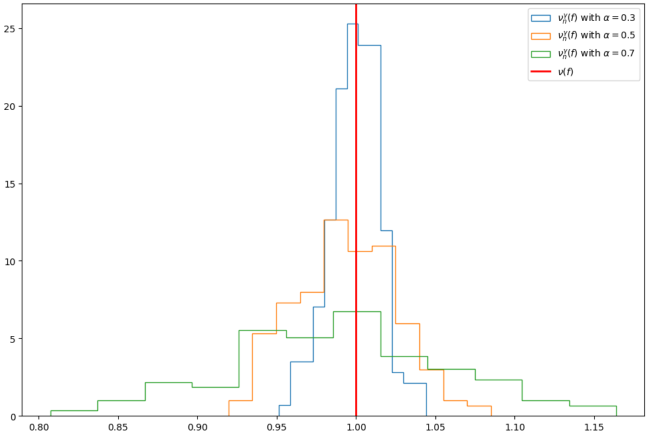


Figure – Histogram of $\nu_n^\gamma(f)$ for different α

Convergnce Figures

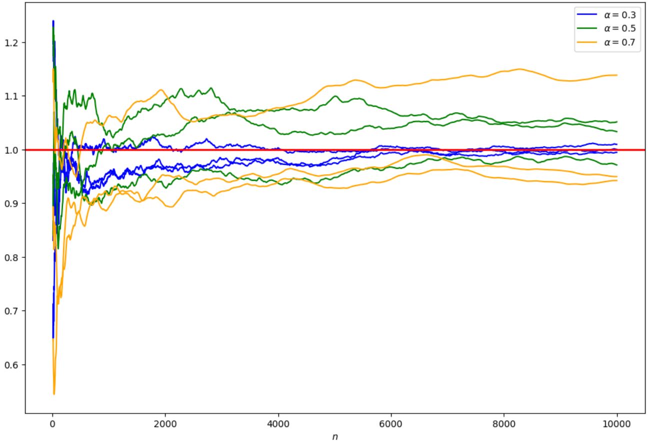


Figure – Values of $\nu_n^\gamma(f)$ for different α

Rate of Convergence

In order to...

- reduce the constant when f has a large growth rate and α is small, one can add n_0 to γ_k, i.e.

$$\gamma_k = (k + n_0)^{-\alpha};$$

- increase the rate of convergence, theoretically, for f of form Ag, the best α is $1/3$.

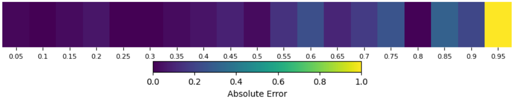


Figure – Normalized absolute errors of $\nu_n^\gamma(f)$ for different α

Recursive Formula of O-U process

Since the Ornstein-Uhlenbeck process has an exact solution, we can verify the result from the previous section :

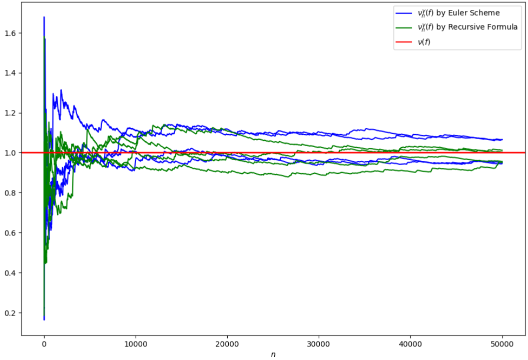


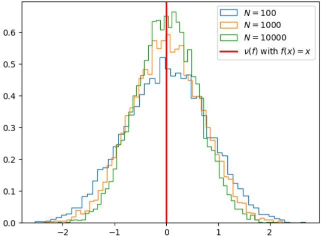
Figure – Values of $\nu_n^\gamma(f)$ for different approach

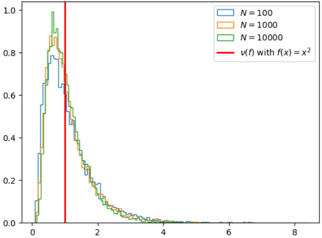
Almost Sure CLT

Let (U_n) be a sequence of i.i.d. square-integrable random variables, satisfying $\mathbb{E}[U_1] = 0$ and $\mathrm{Var}(U_1) = 1$, we have then

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{(U_1+\cdots+U_k)/\sqrt{k}} \to \mathcal{N}(0,1)$$

in distribution almost surely.





Case of Non-Uniqueness

Inspired by ODE $x' = -V'(x)$, we define the following SDE

$$dX_t = -V'(X_t)dt + \sigma dW_t,$$

where σ can be changed.

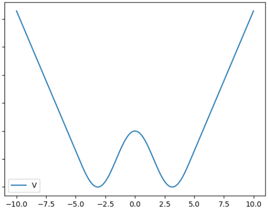


Figure – Image of V

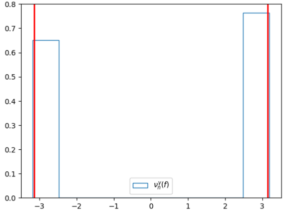


Figure – Case of $\sigma = 0.5$

Its histogram can also be centered Gaussian when σ is big.

Conclusion

Through our efforts in P1 , we ...

- comprehend a series of foundational works ;
- apply key theorems presented to derive new result ;
- verify the result through numerical simulations.

In the P2, we will ...

- consider the rate of convergence from theoretical perspectives ;
- explore more things.